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VOLUME AVERAGING, EFFECTIVE STRESS RULES, AND INVERSION FOR MICROSTRUCTURAL RESPONSE OF MULTICOMPONENT POROUS MEDIA

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Abstract—A general volume-averaging technique is used to derive equations satisfied by the average scalar stresses and strains in multicomponent porous rock. The resulting equations are combined with general thought experiments to produce the effective-stress rules that determine the volumetric changes of the rock induced by changes in the confining and fluid pressures. The composite porous material specifically treated is an isotropic mixture of two Gassmann materials. Two distinct cases are considered depending on whether the grains at the interface between the Gassmann materials are either (1) welded together (no "cracks" can open between the two constituents) or (2) nonwelded (cracks can open). The effective-stress laws determine not only the overall volumetric changes of a given sample (i.e., changes in sample volume, total pore volume, and fluid-mass content), but determine as well the changes within each Gassmann component individually. This additional level of detail achieved in the analysis is referred to as inversion for the microstructural response. In the nonwelded case, the effective-stress law relating the variation of crack porosity with macroscopic changes in confining and fluid stress can be used to determine optimum strategies for increasing fracture/crack porosity with applications to reservoir production analysis. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Real poroelastic media occurring in nature tend to be very complex materials such as rocks containing water, oil, and/or gas. These materials are inherently heterogeneous composites simply because the presence of a solid/fluid mixture, and the random placement of the fluids in the pore space. However, they will often be multicomponent composites even if the saturating fluid is uniform, because the solid constituents themselves may vary from quartz to shale to clay, etc. Whereas in elastic composites the equations of elasticity are well accepted and the only issue to be addressed in research on such composites is the effective elastic constants themselves, the situation is more complicated in such mixed media. Because these physical systems exhibit a richer variety of behavior, different "effective" equations must be used to describe their reactions to external stimuli. Thus, some fraction of the effort in the field of poroelasticity continues to be devoted to derivation and justification of the equations of motion themselves. This paper addresses this issue from the point of view of multiple solid component poroelastic media.

Recent work by Pride *et al.* (1992) using volume averaging to derive the form of the equations of motion for sound traveling through a fluid-saturated porous medium (Biot, 1962) has been restricted by the assumption that the solid part of the solid/fluid composite was microhomogeneous, i.e., composed of only a single type of solid constituent. We call

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such single-solid porous media "Gassmann materials" in honor of the famous paper by Gassmann (1951), in which he first derived some of the basic and best known results about such poroelastic media. In the present paper, we show how to generalize the results of Pride *et al.* (1992) from Gassmann materials to multicomponent and solid frames and then make use of these results to establish general effective-stress rules for the behavior of such media. Effective-stress rules (Nur and Byerlee, 1971; Berryman, 1992, 1993, 1997) determine the relative importance of confining stress and internal pore-fluid pressure on the response of rock to changes in the state of stress. By eliminating the restriction to single solid component media, we include models that are more representative of real porous materials and rocks.

Some of the earlier approaches to volume averaging include those of Hill (1963), Slattery (1967), Whittaker (1969), Burridge and Keller (1981), and Mei and Auriault (1989)—among many others. The approach of Pride *et al.* (1992), which is also the approach taken here, has much in common with the methods of Slattery and Whittaker. However, these two authors were studying fluid flow through a rigid solid matrix, whereas the present approach necessarily includes the effects of solid deformation. Hill (1963), Burridge and Keller (1981), Whittaker (1986), and Mei and Auriault (1989) have also treated deformable media. One of the major differences between the present approach and these related methods is that we make no assumption of periodicity in the medium and we make relatively few approximations, letting the volume averaging do most of the work. The cell problems that arise in homogenization methods based on periodicity assumptions are replaced here by thought experiments and/or real laboratory experiments to evaluate the unknown coefficients that necessarily arise in all methods.

There has also been some recent and related work on double-porosity, dual-permeability media by Berryman and Wang (1995), Tuncay and Corapcioglu (1995), and Wang and Berryman (1996). Tuncay and Corapcioglu (1995) use a volume averaging approach that is basically the same as that of Pride *et al.* (1992) to analyze the situation in which two distinct types of porosity are present in the system: one due to fractures or cracks and the other due to matrix porosity. Berryman and Wang (1995) and Wang and Berryman (1996) obtain virtually identical results by analyzing the constitutive equations and comparing them with quantities measurable in the laboratory. We will find here that combinations of volume averaging and constitutive equations analysis are required to solve the complex problems we address.

It will prove important to make connection with the definitions of Brown and Korringa (1975), which are themselves based on the well-known jacketed and unjacketed thought experiments of Biot and Willis (1957). Generally similar ideas have also been presented by Rice (1975) and Rice and Cleary (1976), but—for our present purposes—we prefer the more detailed discussion presented by Brown and Korringa. If the total volume of the porous sample is V and the pore volume contained in that sample is V_ϕ (where the porosity is given by $\phi = V_\phi/V$), then Brown and Korringa (1975) define constants so that

$$-\frac{\delta V}{V} = \frac{\delta p_{\mathrm{d}}}{K^*} + \frac{\delta p_{\mathrm{f}}}{K_{\mathrm{s}}} \tag{1}$$

and

$$-\frac{\delta V_\phi}{V_\phi} = \frac{\delta p_{\mathrm{d}}}{K_{\mathrm{p}}} + \frac{\delta p_{\mathrm{f}}}{K_\phi}. \tag{2}$$

The independent variables in these formulas are the changes in differential pressure δp_{d} and pore-fluid pressure δp_{f}. The differential pressure is the difference between the external (confining) pressure δp_{c} and the fluid pressure, so $\delta p_{\mathrm{d}} = \delta p_{\mathrm{c}} - \delta p_{\mathrm{f}}$. The coefficients are written in terms of the frame (or jacketed) bulk modulus K^*, the solid grain (or unjacketed) bulk modulus K_{s}, and the pore (or unjacketed pore) bulk modulus K_ϕ. The remaining modulus K_{p} can be shown to be related to K^*, K_{s}, and the porosity ϕ by the formula

$$\frac{\phi}{K_p} = \frac{1}{K^*} - \frac{1}{K_s}, \tag{3}$$

assuming only than an energy density for the bulk deformations exists. Measurements of K_p have been made by Zimmerman *et al.* (1986) for some rocks. One other important factor is that, if the porous solid frame is composed of a single constituent (microhomogeneity), then—and only then—$K_\phi = K_s = K_m$, where K_m is the bulk modulus of the single type of mineral grain present. These definitions and results were used extensively by Berryman (1992) to establish general effective-stress rules for multicomponent rock.

The next section of the paper presents a brief review of the main results of Pride *et al.* (1992), and relates those results to the definitions of Brown and Korringa (1975). Subsequent sections of the paper describe our results on rock models with multiple solid constituents. Section 3 introduces models with two constituents and addresses both welded and nonwelded contacts. Section 4 shows how to write the general effective stress for materials with welded contact. Section 5 summarizes the main conclusions. A series of appendices expands on some of the details of the analysis left out of the main text.

For simplicity and to allow us to focus on the main ideas, the analysis in the paper is restricted to statistically isotropic composite poroelastic media containing isotropic solid constituents.

2. REVIEW OF SINGLE SOLID COMPONENT MODEL AND RESULTS

The issue addressed by Pride *et al.* (1992) concerns the method by which one arrives at the equations of motion for sound traveling through a solid/fluid mixture when it is assumed that the solid is cohesive and porous, but contains only a single type of mineral. In the present section and throughout the paper, the fluid is homogeneous and completely fills the pores.

2.1. *The averaging theorem*

The averaging theorem to be used is due to Slattery (1967) and is based on the idea that volume averages of derivatives are closely related to derivatives of volume averages, but care must be taken to account properly for behavior of the averaged quantities at points or surfaces where abrupt changes occur. In particular, when the quantity to be averaged exists on one side of an interface and does not exist on the other side, an interior interface term will contribute to the volume average of the derivative, but not to the derivative of the volume average.

Suppose that Q is a quantity to be averaged; Q can be a scalar, vector, or tensor. For convenience of the discussion, we will assume that the averaging volume is a finite sphere centered at position $\mathbf{x}$, although other choices are also possible. We label this volume $\Omega(\mathbf{x})$ and the surface of this volume is $\partial\Omega$. The exterior surface has two parts $\partial\Omega = \partial E_0 + \partial E_Q$, with ∂E_0 being the part where the quantity of interest Q vanishes identically and ∂E_Q being the part where $Q \neq 0$. In addition to the exterior surface, there is also an interior surface where Q changes abruptly to zero (or is simply undefined, such as the value of the solid displacement in the interior of a pore) and we label this surface ∂I_Q, for interior. The exterior surface together with this interior surface is the total bounding surface for the region we will label Ω_Q, i.e., the region wherein the quantity to be averaged Q is nonzero. With these definitions, it is straightforward to show that

$$\int_\Omega \nabla Q \, \mathrm{d}^3x = \int_{\Omega_Q} \nabla Q \, \mathrm{d}^3x = \int_{\partial E_Q} \hat{\mathbf{n}}_Q Q \, \mathrm{d}S + \int_{\partial I_Q} \hat{\mathbf{n}}_Q Q \, \mathrm{d}S, \tag{4}$$

where $\mathrm{d}S$ is the infinitesimal of the surface volume element, and $\hat{\mathbf{n}}_Q$ is the unit outward normal vector from the region containing nonzero Q. The main point of (4) is just that $\partial E_Q + \partial I_Q$ is the entire bounding surface of Q in the volume Ω. As an example of the meaning of this result, consider Q to be a vector quantity, take the trace of (4), and the result is just a statement of the well-known divergence theorem for vectors.

The second result is that

$$\nabla\int_{\Omega} Q\,\mathrm{d}^3x = \nabla\int_{\Omega_Q} Q\,\mathrm{d}^3x = \int_{\partial E_Q} \hat{\mathbf{n}}_Q Q\,\mathrm{d}S. \tag{5}$$

The resulting Frechet derivative (5) follows from the fact that the volumes $\Omega(\mathbf{x})$ and $\Omega(\mathbf{x}+\delta\mathbf{x})$ contain virtually the same internal surface and so these do not contribute to the gradient, only the surface average of Q over E_Q matters (for example, the hatched areas in Fig. 1).

Combining these results finally gives

$$\int_{\partial E_Q} \hat{\mathbf{n}}_Q Q\,\mathrm{d}S = \nabla\int_{\Omega} Q\,\mathrm{d}^3x = \int_{\Omega} \nabla Q\,\mathrm{d}^3x - \int_{\partial I_Q} \hat{\mathbf{n}}_Q Q\,\mathrm{d}S. \tag{6}$$

Dividing by the fixed initial volume $V = \int_\Omega \mathrm{d}^3x$ contained in Ω and defining the average as $\langle Q\rangle = V^{-1}\int_\Omega Q\,\mathrm{d}^3x$, then gives the averaging theorem:

$$\nabla\langle Q\rangle = \langle\nabla Q\rangle - \frac{1}{V}\int_{\partial I_Q} \hat{\mathbf{n}}_Q Q\,\mathrm{d}S. \tag{7}$$

One further definition is required to understand the notation to be used for the single solid analysis. The average $\langle Q\rangle$ is an average over the whole volume of Ω (i.e. including

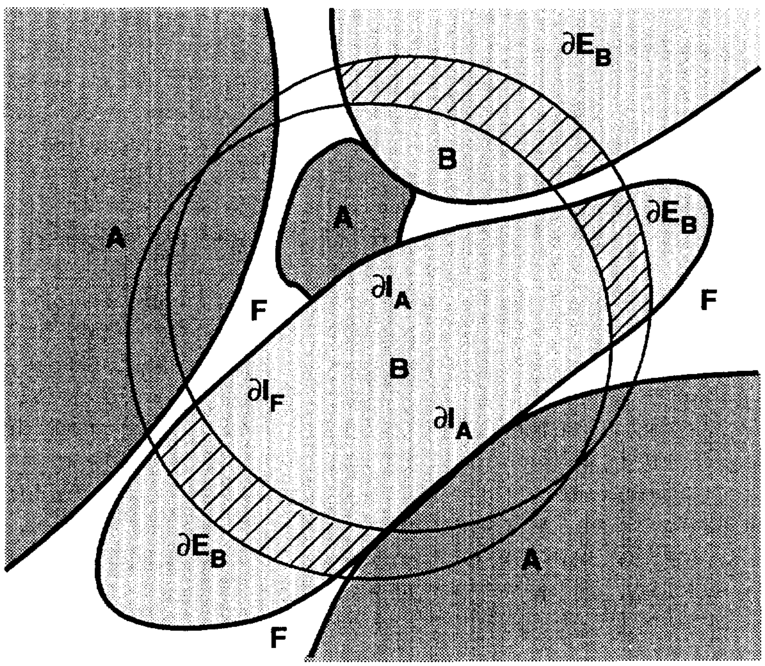

Fig. 1. Schematic illustration for averaging for solid material B in an inhomogeneous fluid/solid mixture containing solids of type-A and B and also containing saturating fluid F. When the averaging sphere (circle with lighter line) is displaced a very small distance, it produces contributions to the Frechet derivative of physical quantities associated by B material along the external B interfaces (labeled by ∂E_B). The internal interfaces ($\partial I_A, \partial I_F$) do not contribute because, as the sphere displacement becomes infinitesimally small, the interfaces contained in both the original sphere and in the displaced sphere are virtually identical. For the volume averaging step alone, no restrictions to Gassmann porous media needs to be made in the analysis. That restriction arises in the thought experiments required to close the system of equations.

the regions where Q is zero by definition), while we will also want to consider the partial average $\bar{Q}$, which are related to the full volume average by

$$\langle Q\rangle = \bar{v}_Q\bar{Q}. \tag{8}$$

In eqn (8), $\bar{v}_Q$ is the volume fraction of Ω in which Q is nonzero.

Note that, although we generally neglect to show this dependence, all the average quantities are in fact functions of the particular choice of averaging volume $\Omega(\mathbf{x})$. In principle, $\Omega(\mathbf{x})$ can be as large as the sample being studied, or as small as desired. The legitimacy of the averaging theorem does not depend on the size of the averaging volume. However, some intermediate choice will generally be made for $\Omega(\mathbf{x})$. Too small of an averaging volume implies rapid fluctuations in the quantities of interest (like the fluid and solid dilatations), while a very large averaging volume implies all the coefficients in the equations are universal constants and therefore prevents us from studying the effect of local inhomogeneities, which is the main purpose of this paper. Also, for a wave problem, when the averaging volume becomes as large as the wavelengths, the oscillatory changes in particle displacement will tend to average to zero. Further discussion of the averaging is presented by Pride and Berryman (1998).

2.2. *Quasistatic constitutive relations for isotropic materials*

Pride *et al.* (1992) perform bulk averages on the microscopic stress/strain relations for a fluid/solid mixture; then, using the averaging theorem, they obtain general constitutive relations for the solid and fluid stress tensors. Taking the trace of these equations gives the following results. The constitutive relations for dilatations and porosity are

$$-\frac{\delta p_s}{K_s} = \nabla\cdot\bar{\mathbf{u}}_s - \frac{\delta\phi}{1-\phi} \tag{9}$$

and

$$-\frac{\delta p_f}{K_f} = \nabla\cdot\bar{\mathbf{u}}_f + \frac{\delta\phi}{\phi}, \tag{10}$$

where the partial averages $\bar{\mathbf{u}}_s$ and $\bar{\mathbf{u}}_f$ are related to the full volume averages by $\langle\mathbf{u}_s\rangle = (1-\phi)\bar{\mathbf{u}}_s$ and $\langle\mathbf{u}_f\rangle = \phi\bar{\mathbf{u}}_f$. One assumption implicit in (9) and (10) is that ϕ changes much more slowly in space than the displacement variables $\bar{\mathbf{u}}_s$ and $\bar{\mathbf{u}}_f$. This assumption allows us to remove the factors involving the porosity from the divergence terms. Although this assumption is nonessential [Pride and Berryman (1998) derive the general poroelasticity laws without using this assumption], it is expedient for our present purposes.

It is important now to understand the interpretations of all the symbols appearing in these two equations [Pride and Berryman (1998) derive the exact interpretations]. First, the variable δp_f is just the change in the average fluid pressure throughout the fluid phase. The change in average solid pressure δp_s is related to the macroscopic confining pressure change δp_c by the averaging relation $\delta p_c = (1-\phi)\delta p_s + \phi\delta p_f$. Thus, δp_s is just the average change in solid pressure experienced by the solid. We view δp_c and δp_f as the pressures we can control (i.e., the independent variables), while the derived quantity (and therefore dependent variable) δp_s is the (solid volume) weighted average of the confining pressure after subtracting that part of the confining pressure supported by the fluid pressure. The change in porosity is given by $\delta\phi$. The porosity change occurs naturally in these expressions because (see Appendix A)

$$\delta\phi = \frac{1}{V}\int_{\partial I_f}\hat{\mathbf{n}}_f\cdot\mathbf{u}_f\,\mathrm{d}S = -\frac{1}{V}\int_{\partial I_s}\hat{\mathbf{n}}_s\cdot\mathbf{u}_s\,\mathrm{d}S. \tag{11}$$

The divergence of the average solid displacement $\nabla\cdot\bar{\mathbf{u}}_s$ is properly interpreted as the dilatation of the porous solid frame (not the dilatation of the solid alone). This interpretation is not obvious, but it follows from the fact that the term arises from the external surface integral [cf eqn (5)]

$$\nabla \cdot \langle \mathbf{u}_s \rangle = \frac{1}{V} \int_{\partial E_s} \hat{\mathbf{n}}_s \cdot \mathbf{u}_s \, \mathrm{d}S, \tag{12}$$

which is exactly the surface integral needed to define the overall behavior of the porous solid frame. Thus, in terms of the definitions of Brown and Korringa (1975),

$$\nabla \cdot \bar{u}_s = \frac{\delta V}{V} = e = -\frac{\delta p_d}{K^*} - \frac{\delta p_f}{K_s}. \tag{13}$$

This interpretation is the same one obtained by Pride *et al.* (1992) using a combination of the standard thought experiments (jacketed and unjacketed) of Biot and Willis (1957). To check that this is so, we can easily show that

$$\frac{\delta\phi}{1-\phi} = -\left[\frac{1-\phi}{K^*} - \frac{1}{K_s}\right](\delta p_s - \delta p_f), \tag{14}$$

using either approach when a single constituent is present so that the Brown and Korringa unjacketed constants satisfy $K_s = K_\phi$, as has been assumed for a single solid constituent. Note that (14) can also be written as

$$\delta\phi = -\left[\frac{1-\phi}{K^*} - \frac{1}{K_s}\right]\delta p_d, \tag{15}$$

emphasizing that porosity is constant if differential pressure is constant—a general result for microhomogeneous porous frames, but not true otherwise. Thus, the left-hand side of (9) is just the solid dilatation $\delta V_s/V_s$, while the two terms on the right-hand are $\delta V/V + \delta(1-\phi)/(1-\phi)$.

Similarly, it is important to understand that the expression $\nabla \cdot \bar{\mathbf{u}}_f$ is not just a fluid dilatation, but also includes the effects of fluid displacement into and out of the volume. In fact, this is already apparent from (10) since the strict fluid dilatation satisfies

$$-\frac{\delta V_f}{V_f} = \frac{\delta p_f}{K_f}, \tag{16}$$

yet (10) contains an additional term related to changes in porosity. The correct physical interpretation of $\nabla \cdot \bar{\mathbf{u}}_f$ is provided by its relation to the increment of fluid content

$$\zeta = \phi(\nabla \cdot \bar{\mathbf{u}}_s - \nabla \cdot \bar{\mathbf{u}}_f), \tag{17}$$

where ζ is defined as

$$\zeta = \frac{\delta V_\phi - \delta V_f}{V} = \phi\left(\frac{\delta V}{V} - \frac{\delta V_f}{V_f}\right) + \delta\phi \tag{18}$$

and has the interpretation (Biot, 1973; Berryman and Thigpen, 1985) of the relative change in fluid mass per unit volume of initial fluid mass. Note that (16) and (18) are in agreement with (17) if the averaging equation (10) is also satisfied.

The equations (9), (10), and (13) are sufficient to arrive at the standard form of the equations relating to e and ζ to the macroscopic pressures δp_c and δp_f for a single constituent porous medium given by

$$\begin{pmatrix} e \\ -\zeta \end{pmatrix} = \begin{pmatrix} 1/K^* & 1/K_s - 1/K^* \\ 1/K_s - 1/K^* & 1/K^* + \phi/K_f - (1+\phi)/K_s \end{pmatrix} \begin{pmatrix} -\delta p_c \\ -\delta p_f \end{pmatrix}. \tag{19}$$

These equations are completely consistent with the results of Pride *et al.* (1992) as can be demonstrated by substituting the definitions given above into the formulas (48) and (49) of Pride *et al.* (1992), and then doing a straightforward (though somewhat tedious) 2×2 matrix inversion.

3. MODELS WITH TWO SOLID COMPONENTS

We now consider that the porous sample is a mixture of two different porous materials denoted by A and B. (See Fig. 1.) Both of these A and B components are taken as Gassmann materials and are assumed to occupy well defined (distinct) volumes within the sample so that $V_A + V_B = V$ where V is the total sample (or averaging) volume. The A and B components are sufficiently uniform and of sufficient extent that they can be thought of as homogeneous porous continua. The volume fractions $v_A = V_A/V$ and $v_B = V_B/V$ are assumed to be known as are the standard poroelastic properties of the two Gassmann materials: these are the porosities (ϕ_A, ϕ_B), the drained bulk moduli (K_A^*, K_B^*), and the bulk moduli of the A and B minerals (K_A, K_B). A single homogeneous fluid is assumed to saturate both the A and B components and has a known bulk modulus K_f. Finally, the drained bulk modulus of the total composite sample K^* is assumed to be known as well.

If increments in confining pressure and fluid pressure are applied to the sample as a whole, the central goal in what follows is to define the various volumetric responses of the sample in terms of the known material properties defined above. For the sample as a whole, these responses are determined by three Brown and Korringa moduli (K^*, K_s, and K_ϕ) where K^* is known but K_s and K_ϕ must be determined. When the sample is a mixture of two isotropic Gassmann components, Berryman and Milton (1991, 1992) have introduced uniform strain thought experiments that allow K_s and K_ϕ to be solved for exactly. We show here that the volumetric changes of the individual A and B components can also be exactly determined using the same known properties used to define the overall response. This is what we mean by "inversion for the microstructural response" and is something that has not previously been addressed in the literature.

In fact, we go further and distinguish two cases: (1) the interface between the A and B Gassmann components within the sample is welded and (2) the interface is either only partially welded or nonwelded. The key distinction between the two cases is that welded contact between porous constituents implies that no cracks/fractures can open up between these constituents due to applied temperature or stress, while cracks/fractures are allowed with only partially welded contact. Welded contact may be somewhat easier to analyze, but nonwelded contact is expected to be a better model of rocks (Walsh, 1965). In order to obtain the key results in the case of nonwelded contact, the response due to temperature changes will have to be considered.

As a means of simplifying the algebra in the following analysis, we introduce (see Appendix A for the definition and a discussion) new quantities that we call $\tilde{\mathbf{u}}_A$ and $\tilde{\mathbf{u}}_B$, the divergences of which are just the dilatations of the corresponding Gassmann components: $\nabla \cdot \tilde{\mathbf{u}}_A = \delta V_A / V_A$ and $\nabla \cdot \tilde{\mathbf{u}}_B = \delta V_B / V_B$. This step is made at this point because it helps to avoid introducing various terms that would ultimately cancel in the final formulas.

3.1. *Fully welded contact*

For two porous components A and B in welded contact, the volume fractions of the components v_A, v_B satisfy

$$v_A + v_B = 1, \tag{20}$$

while the overall porosity is given by

$$\phi = v_A\phi_A + v_B\phi_B. \tag{21}$$

The fractions of the total volume occupied by the solid components are $\bar{v}_A = v_A(1-\phi_A)$ and $\bar{v}_B(1-\phi_B)$, respectively. The solid components in A and B are individually pure, so that each porous constituent may be thought of as a microhomogeneous (or Gassmann) material.

3.1.1. *Averaging equations.* Volume averaging for the three components (two solid and one fluid) yields

$$-\frac{\delta p_A}{K_A} = \nabla\cdot\tilde{\mathbf{u}}_A - \frac{\delta\phi_A}{(1-\phi_A)}, \tag{22}$$

$$-\frac{\delta p_B}{K_B} = \nabla\cdot\tilde{\mathbf{u}}_B - \frac{\delta\phi_B}{(1-\phi_B)}, \tag{23}$$

and

$$-\frac{\delta p_f}{K_f} = \nabla\cdot\tilde{\mathbf{u}}_f + \frac{\delta\phi}{\phi}. \tag{24}$$

These three equations should be compared to the single-component results (9) and (10). The pressure increment δp_A represents the average change in solid pressure throughout the grains of the porous material A (with an analogous definition for δp_B). To provide some insight, notice that the left-hand side of (22) is easily seen to be $\delta[V_A(1-\phi_A)]/V_A(1-\phi_A)$ (which is the relative change in type-A solid volume), while the right-hand side is $\delta V_A/V_A - \delta\phi_A/(1-\phi_A)$.

3.1.2. *Other relationships.* It is useful to think of eqns (22) and (23) as equations for the changes in the constituent porosities $\delta\phi_A$ and $\delta\phi_B$. To relate these values to the overall response, we need another pair of equations. First, note that from (21)

$$\delta\phi = v_A\delta\phi_A + v_B\delta\phi_B + \delta v_A(\phi_A - \phi_B), \tag{25}$$

so we need an expression for the change in v_A. For welded contact, we obtain such an expression by noting that by definition

$$v_A + \delta v_A = \frac{V_A(1+\nabla\cdot\tilde{\mathbf{u}}_A)}{V_A(1+\nabla\cdot\tilde{\mathbf{u}}_A) + V_B(1+\nabla\cdot\tilde{\mathbf{u}}_B)}, \tag{26}$$

which upon expansion and neglect of second order terms yields

$$\delta v_A = v_A v_B(\nabla\cdot\tilde{\mathbf{u}}_A - \nabla\cdot\tilde{\mathbf{u}}_B), \tag{27}$$

and furthermore for welded contact $\delta v_B = -\delta v_A$. Note that, if A and B expand or contract at the same rate so that $\nabla\cdot\tilde{\mathbf{u}}_A = \nabla\cdot\tilde{\mathbf{u}}_B$, then $\delta v_A = 0$ as expected.

We also want to view the combined solid volume $V_s = V_A(1-\phi_A) + V_B(1-\phi_B)$ as a whole in order to recover Biot's (or Brown and Korringa's) macroscopic equations for the inhomogeneous material. Then, it is important to recognize that the solid dilatations must satisfy

$$\nabla \cdot \bar{\mathbf{u}}_s = v_A \nabla \cdot \tilde{\mathbf{u}}_A + v_B \nabla \cdot \tilde{\mathbf{u}}_B, \tag{28}$$

and, similarly, the solid pressures must satisfy

$$(1-\phi)\delta p_s = \bar{v}_A \delta p_A + \bar{v}_B \delta p_B. \tag{29}$$

Relation (28) may be easily derived by considering the denominator of the right-hand side of (26), whereas (29) is just a statement of force conservation across a material boundary. Alternatively stated, both (28) and (29) are direct consequences of the definitions of the volume averages.

3.1.3. *Definitions of microstructural coefficients.* Now the issue to be addressed is how the volume, porosity, and average solid pressure of the individual A and B porous materials change when increments in confining and fluid pressure are applied to the sample as a whole. We assume the system is isotropic and linear, so all variables can be assumed to depend linearly on small changes in the applied stresses. Thus, the changes of the average solid pressure within the A and B components are taken to be linear functions of the form

$$(1-\phi_A)\delta p_A = Q_A(\delta p_c - \gamma_A \delta p_f) \tag{30}$$

and

$$(1-\phi_B)\delta p_B = Q_B(\delta p_c - \gamma_B \delta p_f). \tag{31}$$

Due to the effect of microstructure, these average solid pressures will differ, in general, from the average solid pressure throughout the sample as a whole. Expressing Q_A, Q_B, γ_A and γ_B in terms of the Brown and Korringa moduli and other known properties is really the key to this analysis because it will be possible to express the volumetric response of the A and B components entirely in terms of them and the known properties of the Gassmann materials.

The A and B dilatations are defined by linear response laws of the form

$$-\nabla \cdot \tilde{\mathbf{u}}_A = S_A(\delta p_c - \bar{\alpha}_A \delta p_f) \tag{32}$$

and

$$-\nabla \cdot \tilde{\mathbf{u}}_B = S_B(\delta p_c - \bar{\alpha}_B \delta p_f), \tag{33}$$

while the A and B porosity changes are defined by

$$-\delta\phi_A = W_A(\delta p_c - \chi_A \delta p_f) \tag{34}$$

and

$$-\delta\phi_B = W_B(\delta p_c - \chi_B \delta p_f). \tag{35}$$

Such linear response is always expected (Hill, 1963; Berryman and Berge, 1996), but not always possible to determine explicitly in inhomogeneous systems. Here S_A and S_B, for example, are constant compliances, while $\bar{\alpha}_A$ and $\bar{\alpha}_B$ are effective stress coefficients—analogous to the Biot–Willis parameter but differing from it in general because of the effects of microgeometry in the composite; i.e., because the average pressure increments throughout the A and B components will be different, in general, from the confining pressure δp_c applied to the sample as a whole. We will call constants such as S_A, S_B, $\bar{\alpha}_A$, and $\bar{\alpha}_B$, and the analogous quantities for pressure and porosity changes the "microstructural response coefficients".

There are 12 such microstructural coefficients to be determined (Q_A, Q_B, γ_A, γ_B, S_A, S_B, $\bar{\alpha}_A$, $\bar{\alpha}_B$, W_A, W_B, χ_A, χ_B). Our strategy in what follows is to: (1) define the A and B micro-pressure coefficients in terms of the Brown and Korringa moduli (K^*, K_s, K_ϕ) and other known constants by combining the volume-averaging results with the overall response laws of Brown and Korringa; (2) introduce the known micro-pressure laws into the A and B Gassmann laws in order to define the coefficients in the dilatation and porosity response laws of the A and B components; and, lastly (3) define the moduli K_s and K_ϕ by means of the Berryman and Milton (1991) thought experiment.

3.1.4. *General relations for microstructural response.* We now use the averaging equations to express the micro-pressure coefficients in terms of the Brown and Korringa moduli. Equation (29) shows that

$$v_A Q_A(\delta p_c - \gamma_A \delta p_f) + v_B Q_B(\delta p_c - \gamma_B \delta p_f) = \delta p_c - \phi \delta p_f. \tag{36}$$

This equality is completely general for arbitrary values of δp_c and δp_f, which implies that the coefficients of each pressure change must satisfy the equality separately (first set $\delta p_f = 0$, then set $\delta p_c = 0$). The resulting equations are

$$v_A Q_A + v_B Q_B = 1 \tag{37}$$

and

$$v_A Q_A \gamma_A + v_B Q_B \gamma_B = \phi. \tag{38}$$

Equations (37) and (38) give us two equations for these four unknown coefficients.

To obtain more equations relating the coefficients to known quantities, we consider the volume averaging eqns (22) and (23). Multiplying (22) by $\bar{v}_A$ and (23) by $\bar{v}_B$, adding, and substituting (25) and (28), we have the equation relating the constituent changes in average pressure to the overall solid dilatation

$$-\bar{v}_A \frac{\delta p_A}{K_A} - \bar{v}_B \frac{\delta p_B}{K_B} = (1-\phi)\nabla \cdot \bar{\mathbf{u}}_s - \delta\phi. \tag{39}$$

The right hand side of (39) can be written in terms of macroscopic coefficients by recalling that the definitions of Brown and Korringa show the porosity change is of the form $\delta(V_\phi / V)$, which is

$$-\delta\phi = \frac{\alpha - \phi}{K^*}(\delta p_c - \delta p_f) + \phi\left(\frac{1}{K_\phi} - \frac{1}{K_s}\right)\delta p_f. \tag{40}$$

Thus, using (13) and (40), we find that

$$-\bar{v}_A \frac{\delta p_A}{K_A} - \bar{v}_B \frac{\delta p_B}{K_B} = -\frac{\delta p_c}{K_s} + \frac{\phi \delta p_f}{K_\phi}. \tag{41}$$

This equation leads to two more equations for the Qs and the γs:

$$\frac{v_A Q_A}{K_A} + \frac{v_B Q_B}{K_B} = \frac{1}{K_s}, \tag{42}$$

and

$$\frac{v_A Q_A \gamma_A}{K_A} + \frac{v_B Q_B \gamma_B}{K_B} = \frac{\phi}{K_\phi}. \tag{43}$$

Solving the pair of eqns (37) and (42) for the Qs gives

$$\begin{pmatrix} v_A Q_A \\ v_B Q_B \end{pmatrix} = \frac{1}{1/K_B - 1/K_A} \begin{pmatrix} 1/K_B - 1/K_s \\ 1/K_s - 1/K_A \end{pmatrix} \tag{44}$$

and solving the pair of eqns (38) and (43) for the γs gives

$$\begin{pmatrix} v_A Q_A \gamma_A \\ v_B Q_B \gamma_B \end{pmatrix} = \frac{\phi}{1/K_B - 1/K_A} \begin{pmatrix} 1/K_B - 1/K_\phi \\ 1/K_\phi - 1/K_A \end{pmatrix}, \tag{45}$$

showing that

$$\gamma_A = \phi \frac{1/K_B - 1/K_\phi}{1/K_B - 1/K_s} \tag{46}$$

with a similar expression for γ_B. Since $K_\phi \neq K_s$ in general, we see that both γs differ somewhat from ϕ. Thus, the average pressure response of the solid components due to changes in external pressure has been completely expressed in terms of the Brown and Korringa moduli. (We will find the same result holds as well for nonwelded contact.) These are the key results that when combined with the Gassmann laws of the next section allow all microstructural coefficients to be expressed in terms of K^*, K_s and K_ϕ and known properties of the Gassmann materials.

Although not essential to our general program here, the averaging equations do imply certain other relations involving the microstructure coefficients. For example, (28) shows that for the sample as a whole

$$v_A S_A(\delta p_c - \bar{\alpha}_A \delta p_f) + v_B S_B(\delta p_c - \bar{\alpha}_B \delta p_f) = \frac{1}{K^*}(\delta p_c - \alpha \delta p_f), \tag{47}$$

where $\alpha = 1 - K^*/K_s$ is the Biot–Willis parameter. From (47) it follows that, since δp_c and δp_f are independent variables,

$$v_A S_A + v_B S_B = \frac{1}{K^*} \tag{48}$$

and

$$v_A S_A \bar{\alpha}_A + v_B S_B \bar{\alpha}_B = \frac{\alpha}{K^*}. \tag{49}$$

Equations (48) and (49) give two equations in these four unknowns.

Now to get relations involving the porosity changes, we consider eqns (22) and (23). Substituting from the definitions (32) and (35), we obtain

$$W_A = (1 - \phi_A) S_A - \frac{Q_A}{K_A} \tag{50}$$

and

$$W_A\chi_A = (1-\phi_A)S_A\bar{\alpha}_A - \frac{Q_A\gamma_A}{K_A}, \tag{51}$$

with similar expressions for W_B and χ_B.

Not all of these results can be independent because equation (39) was also obtained by taking a linear combination of (22) and (23). Indeed, substituting (50) and (51) into the porosity eqn (25) just reproduces (48) and (49). Considering (48) and (49), we have one system of two equations in four unknowns. Thus, the implications of the averaging equations alone has brought us to the point where just two more equations will allow all the microstructural coefficients to be expressed in terms of K^*, K_s, and K_ϕ.

3.1.5. *Gassmann laws for components* A *and* B. The central assumptions made about materials A and B to this point have been that they respond linearly and isotropically to the applied pressure increments. Furthermore, in the averaging eqns (22)–(23), we have assumed that the minerals comprising the frames of materials A and B are homogeneous. Thus, even before introducing the Gassmann laws here, the analysis has been restricted to a treatment requiring the A and B components to be isotropic monomineral porous materials (i.e., Gassmann materials).

Pride and Berryman (1998) have shown that the volumetric response of an isotropic (but otherwise arbitrary) porous material does not depend on the details of how stress is applied to the surface of the sample. The volumetric response depends only on the average fluid and solid pressures generated throughout the sample. Since our A and B Gassmann components are each isotropic, this says that even though they are entangled within the sample and have nonuniform stress fields distributed over their surfaces in general, their "isolated state" compressibility laws still hold so long as we use the micro-confining pressures δp_{cA} and δp_{cB} defined as the average total pressures throughout the A and B regions, respectively. Such micro-confining pressures are thus given by

$$\delta p_{cA} = (1-\phi_A)\delta p_A + \phi_A\delta p_f \tag{52}$$

$$= Q_A\delta p_c + (\phi_A - Q_A\gamma_A)\delta p_f \tag{53}$$

where the microstructural pressure law (30) has been used. Analagous expressions hold for δp_{cB}.

To close the system of microstructural coefficients, the Gassmann compressibility laws for the A and B components are now given. Instead of working with these laws in the form of (19), it is more convenient to write them as

$$-\nabla\cdot\tilde{\mathbf{u}}_A = \frac{1}{K_A^*}(\delta p_{cA} - \alpha_A\delta p_f) \tag{54}$$

$$= \frac{Q_A}{K_A^*}\left[\delta p_c - \frac{(\alpha_A - \phi_A + Q_A\gamma_A}{Q_A)}\delta p_f\right] \tag{55}$$

$$-\delta\phi_A = \frac{\alpha_A - \phi_A}{K_A^*}(\delta p_{cA} - \delta p_f) \tag{56}$$

$$= \frac{(\alpha_A - \phi_A)Q_A}{K_A^*}\left[\delta p_c - \frac{(1-\phi_A + Q_A\gamma_A)}{Q_A}\delta p_f\right] \tag{57}$$

where α_A is again the Biot–Willis constant $\alpha_A = 1 - K_A^*/K_A$. Analogous laws hold for the B component.

We now simply read off the correspondence between the microstructural laws and the Gassmann laws (55) and (57) to find

$$S_A = \frac{Q_A}{K_A^*} \tag{58}$$

$$\alpha_A = \gamma_A + \frac{\alpha_A - \phi_A}{Q_A} \tag{59}$$

$$W_A = \frac{(\alpha_A - \phi_A)Q_A}{K_A^*} \tag{60}$$

$$\chi_A = \gamma_A + \frac{1 - \phi_A}{Q_A} \tag{61}$$

with identical expressions for the B coefficients. Since the Qs and γs have already been obtained, we now have all microstructural coefficients uniquely (and exactly) expressed in terms of K^*, K_s, and K_ϕ.

Note that eqns (48)–(51) provide alternative expressions for some of these coefficients. In particular, it is easy to show that

$$S_A = \frac{1}{K_A(\chi_A - \bar{\alpha}_A)} \tag{62}$$

$$\bar{\alpha}_A = \chi_A - \frac{v_A(\chi_A - \chi_B)/K_A}{(\alpha - \chi_B)/K^* + v_A/K_A + v_B/K_B}, \tag{63}$$

with analogous expressions for the B coefficients (found always by interchanging subscripts $A-B$). We now proceed to the final step which is to obtain K_s and K_ϕ in terms of known quantities.

3.1.6. *Uniform expansion/contraction thought experiment.* The well-known jacketed experiment of Biot and Willis (1957) has been implicitly used to define both the overall frame modulus K^* and the frame moduli K_A^* and K_B^* of the two Gassmann components. The unjacketed thought experiment of Biot and Willis (in which pore pressure and confining pressure increments are the same) has also been implicitly used to obtain the Gassmann laws for $\delta\phi_A$ and $\delta\phi_B$. However, the unjacketed experiment is not useful in determining K_ϕ and K_s when the solid framework of grains consists of two (or more) distinct mineral types. Fortunately, Berryman and Milton (1991) have presented a different thought experiment for two-component media that allows both K_s and K_ϕ to be determined exactly. [Norris (1992) has confirmed and extended these results using an analogy between the equations of poroelasticity and thermoelasticity.] Due to the terseness with which Berryman and Milton stated their thought experiment, the key ideas are reviewed here.

Our A and B components can be viewed as porous continua subject to the laws of poroelasticity. Consider a body that is a composite of different porous continua and that is immersed in a reservoir of uniform confining pressure. After a sufficient time, the fluid pressure throughout the body is a constant. Both the equilibrium laws of poroelasticity and the boundary conditions on the body's surface can be satisfied by a stress distribution in which the confining stress at all points of the composite is constant and given by the pressure of the reservoir. However, such a uniform stress field will only be the solution of the actual poroelasticity problem if the corresponding strain compatibility equations are satisfied along with the welded interface conditions. With uniform stress, the strain within each component of the body is also uniform and thus the compatibility equations are satisfied. However, because the strain in different components will, in general, be different, there is no way to keep the interfaces welded except for very special geometric arrangements of the components (e.g., concentric layers). Thus, for an arbitrary distribution of different components, the total stress field will not, in general, be uniform throughout the composite body even though such uniform stress always satisfies both the equilibrium law and the external boundary conditions.

What Berryman and Milton (1991) noticed is that if there are only two isotropic components (A and B), one can always select the confining pressure of the reservoir (with fluid pressure fixed) such that the total strain of the two components is equal; i.e., such that $\delta V_A/V_A = \delta V_B/V_B$. One possible way to achieve this macroscopic condition is if the poroelastic strain at each and every point throughout the A and B continua is constant. Such a uniform strain field guarantees that the welded interface conditions are satisfied and, in so doing, corresponds to a total stress field that is also uniform. Since these uniform strain and stress fields satisfy all conditions for uniqueness of a solution, they must be the actual fields throughout the body. Thus, for arbitrary distributions of the A and B components, one finds that when (and only when) $\nabla \cdot \tilde{\mathbf{u}}_A = \nabla \cdot \tilde{\mathbf{u}}_B$, then $\delta p_{cA} = \delta p_{cB} = \delta p_c$ exactly. Furthermore, under these conditions, the poroelastic stress and strain throughout the body are constant. These are the key results of the Berryman and Milton thought experiment.

The special confining pressure at which the strain becomes everywhere uniform is denoted here as $\delta p_c = \theta \delta p_f$. An expression for θ is obtained by equating the Gassmann laws for $\nabla \cdot \tilde{\mathbf{u}}_A$ and $\nabla \cdot \tilde{\mathbf{u}}_B$ when $\delta p_{cA} = \delta p_{cB} = \delta p_c$ to give

$$\theta = \frac{\alpha_A/K_A^* - \alpha_B/K_B^*}{1/K_A^* - 1/K_B^*}. \tag{64}$$

The constant K_s (or, equivalently, the Biot–Willis constant $\alpha = 1 - K^*/K_s$) is determined from the requirement that $\nabla \cdot \tilde{\mathbf{u}}_s = \nabla \cdot \tilde{\mathbf{u}}_A$ ($= \nabla \cdot \tilde{\mathbf{u}}_B$) when $\delta p_c = \theta \delta p_f$ so that

$$\theta = \frac{\alpha_A/K_A^* - \alpha/K^*}{1/K_A^* - 1/K^*} \left(= \frac{\alpha_B/K_B^* - \alpha/K^*}{1/K_B^* - 1/K^*} \right). \tag{65}$$

The combination of (64) and (65) may appear to be too many equations for the same thing, but in fact any two of the three implies the third, so the count is correct.

The constant K_ϕ is determined by noting that $\delta v_A = 0$ during uniform strain so that (25) becomes

$$\delta\phi = v_A \delta\phi_A + v_B \delta\phi_B \tag{66}$$

$$= -(\theta - 1)\left[\frac{v_A(\alpha_A - \phi_A)}{K_A^*} + \frac{v_B(\alpha_B - \phi_B)}{K_B^*}\right]\delta p_f, \tag{67}$$

where the Gassmann law (60) was used for $\delta\phi_A$ and $\delta\phi_B$ with $\delta p_{cA} = \delta p_{cB} = \theta \delta p_f$. A second expression for $\delta\phi$ is obtained by introducing the uniform strain condition $\delta p_c = \theta \delta p_f$ into (40) to give

$$-\int\phi = \left[\frac{\phi}{K_\phi} + \frac{(\theta - 1)(1 - \phi)}{K^*} - \frac{\phi + \theta - 1}{K_s}\right]\delta p_f. \tag{68}$$

These two expressions for $\delta\phi$ combine to give an exact expression for K_ϕ entirely in terms of known quantities.

Finally, the uniform-strain conditions provide alternative expressions for the S_A and S_B microstructural coefficients

$$S_A = \frac{\theta - \alpha}{K^*(\theta - \bar{\alpha}_A)} \quad \text{and} \quad S_B = \frac{\theta - \alpha}{K^*(\theta - \bar{\alpha}_B)}. \tag{69}$$

Substituting (69) into (48) gives

$$\frac{1}{\theta-\alpha}=\frac{v_A}{\theta-\bar{\alpha}_A}+\frac{v_B}{\theta-\bar{\alpha}_B}, \tag{70}$$

which provides an additional equality (not needed for closure) relating the values of the microstructural effective stress coefficients $\bar{\alpha}_A$ and $\bar{\alpha}_B$. (See also Appendices B, C and D.)

3.1.7. *Summary of welded contact results.* For the reader's convenience, the key results of the above analysis are now summarized. The effective stress-laws governing the overall response of a multicomponent porous rock are (Brown and Korringa, 1975)

$$-\nabla\cdot\bar{\mathbf{u}}_s=\frac{1}{K^*}(\delta p_c-\alpha\delta p_f) \tag{71}$$

$$-\delta\phi=\frac{\alpha-\phi}{K^*}(\delta p_c-\chi\delta p_f) \tag{72}$$

where the effective-stress coefficients α and χ are defined by

$$\alpha=1-K^*/K_s \tag{73}$$

$$\chi=1-\frac{\phi/K_\phi-\phi/K_s}{(1-\phi)/K^*-1/K_s}. \tag{74}$$

For the specific case where the rock is an isotropic mixture of two Gassmann porous materials in welded contact, the Brown and Korringa coefficient K_s is expressed (Berryman and Milton, 1991)

$$\frac{1}{K_s}=\frac{1-\theta}{K^*}+\frac{\theta-\alpha_A}{K_A^*}, \tag{75}$$

while the coefficient K_ϕ can be written (Berryman and Milton, 1991)

$$\frac{\phi}{K_\phi}=\frac{\phi}{K_s}+(\theta-1)\left[\frac{v_A(\alpha_A-\phi_A)}{K_A^*}+\frac{v_B(\alpha_B-\phi_B)}{K_B^*}-\frac{(\alpha-\phi)}{K^*}\right]. \tag{76}$$

In both of these expressions, the coefficient

$$\theta=\frac{\alpha_A/K_A^*-\alpha_B/K_B^*}{1/K_A^*-1/K_B^*} \tag{77}$$

defines the pressure ratio $\delta p_c=\theta\delta p_f$ that results in uniform strain throughout the A and B porous continua.

The microstructural response laws of the Gassmann components are

$$-\nabla\cdot\bar{\mathbf{u}}_A=S_A(\delta p_c-\bar{\alpha}_A\delta p_f) \tag{78}$$

$$-\delta\phi_A=W_A(\delta p_c-\chi_A\delta p_f) \tag{79}$$

$$(1-\phi_A)\delta p_A=Q_A(\delta p_c-\gamma_A\delta p_f) \tag{80}$$

where the coefficients are

$$v_A Q_A = \frac{1/K_B - 1/K_s}{1/K_B - 1/K_A} \tag{81}$$

$$\gamma_A = \frac{\phi/K_B - \phi/K_\phi}{1/K_B - 1/K_s} \tag{82}$$

$$S_A = \frac{Q_A}{K_A^*} \tag{83}$$

$$\bar{\alpha}_A = \gamma_A + \frac{\alpha_A - \phi_A}{Q_A} \tag{84}$$

$$W_A = \frac{(\alpha_A - \phi_A)Q_A}{K_A^*} \tag{85}$$

$$\chi_A = \gamma_A + \frac{1-\phi_A}{Q_A}. \tag{86}$$

Similar expressions for the B component response are obtained by replacing the subscript A with B throughout these expressions. Alternative expressions for these coefficients are presented in Appendices B and C as well as in the previous sections.

To summarize the approach, we have: (1) averaged the fluid and solid response throughout the two component body; (2) combined the averaging equations with the Brown and Korringa laws to obtain the micropressure laws for δp_A and δp_B; (3) introduced the micropressure laws into the A and B Gassmann laws in order to identify the other microcoefficients; and (4) used the Berryman and Milton (1991) uniform strain thought experiment in order to find the moduli K_s and K_ϕ.

3.2. *Partially welded or nonwelded contact*

For two porous components A and B in partially welded or nonwelded contact, the volume fractions of the components v_A, v_B satisfy

$$v_A + v_B + v_C = 1, \tag{87}$$

where v_C is the crack or fracture volume fraction which may possibly be zero prior to the application of a change in confining or pore pressure. The porosity is now given by

$$\phi = v_A\phi_A + v_B\phi_B + v_C, \tag{88}$$

where ϕ_A, ϕ_B are again, respectively, the porosities of the porous constituents A and B, but now a third crack (or fracture) phase is present, and being pure void it therefore contributes its entire volume fraction v_C to the porosity.

In order to close the system of equations for a composite porous medium including cracks, we need three applied fields [this fact is clear from the earlier work of Berryman and Milton (1992)] and therefore must generalize the equations of Brown and Korringa (1975) to include the effects of a temperature field. The resulting equations are:

$$-\frac{\delta V}{V} = \frac{\delta p_d}{K^*} + \frac{\delta p_f}{K_s} - \beta^*\delta T, \tag{89}$$

$$-\frac{\delta V_\phi}{V_\phi} = \frac{\delta p_d}{K_p} + \frac{\delta p_f}{K_\phi} - \beta_\phi\delta T, \tag{90}$$

and

$$-\frac{\delta V_{\mathrm{f}}}{V_{\mathrm{f}}}=\frac{\delta p_{\mathrm{f}}}{K_{\mathrm{f}}}-\beta_{\mathrm{f}}\delta T. \tag{91}$$

The new coefficients are the volume thermal expansions β^*, β_ϕ, and β_{f} for the overall composite volume, the pore volume, and the pore fluid, respectively. The change in temperature (assumed uniform throughout the solid and the fluid) is δT. The constant β_ϕ has also been introduced previously by McTigue (1986).

A similar set of equations is valid for each porous constituent with the exception that the unjacketed bulk moduli ($K_{\mathrm{s}} = K_\phi = K_{\mathrm{m}}$) are equal for Gassmann materials, as are the overall volume and pore volume thermal expansions ($\beta^* = \beta_\phi = \beta_{\mathrm{m}}$) [and, of course, the differential pressures must be expressed in terms of the micro-confining pressures (e.g., $\delta p_{cA}-\delta p_{\mathrm{f}}$) when considering the Gassmann material embedded within the A–B composite]. Thus, each Gassmann material has only one thermal expansion parameter associated with it, labeled, respectively, β_A and β_B.

We assume that β_A, β_B, and β^* are all known from experiments. We will find that in order to close the system of coefficients, either β_ϕ or K_ϕ must also be measured. With external stress constant, if the increment in fluid content ζ_{p} due to a temperature deviation δT is measured, then β_ϕ is given as

$$\beta_\phi=\beta_{\mathrm{f}}+\frac{\zeta_{\mathrm{p}}}{\phi\delta T}. \tag{92}$$

For example, if a jacketed sample with a tube piercing the jacket is immersed in a temperature reservoir, β_ϕ can be measured by monitoring the fluid volume changes in the tube. If the tube is made of material with a coefficient of thermal expansion that is dramatically lower than the sample, this measurement may be easier to make than the equivalent measurement of K_ϕ. In what follows, β_ϕ will be considered as known *a priori* and K_ϕ as unknown.

3.2.1. *Volume averaging with temperature changes.* The volume averaging results for the constituents are easily generalized from (22)–(24) to

$$-\frac{\delta p_A}{K_A}+\beta_A\delta T=\nabla\cdot\tilde{\mathbf{u}}_A-\frac{\delta\phi_A}{(1-\phi_A)}, \tag{93}$$

$$-\frac{\delta p_B}{K_B}+\beta_B\delta T=\nabla\cdot\tilde{\mathbf{u}}_B-\frac{\delta\phi_B}{(1-\phi_B)}, \tag{94}$$

and

$$-\frac{\delta p_{\mathrm{f}}}{K_{\mathrm{f}}}+\beta_{\mathrm{f}}\delta T=\nabla\cdot\tilde{\mathbf{u}}_{\mathrm{f}}+\frac{\delta\phi}{\phi}. \tag{95}$$

The meanings of all the constants are the same as in (22)–(24), but the thermal expansion terms have been added to the left-hand side.

We see that the basic averaging equations for partially welded or nonwelded contact are almost the same as (22)–(24) for welded contact, but now temperature changes are included and the interpretation of some terms is a little different. The porous solid volumes V_A and V_B do not complete the whole averaging volume V, so $\delta\phi$ may now include some void space (due to cracks or fractures) outside of these volumes but still inside V. Also, we will find that the average pressure changes (δp_A and δp_B) are now functions of temperature change as well as being functions of both confining and fluid pressure changes.

3.2.2. *Other relations*. Equations (93)–(95) do not explicitly show $\delta\phi_C$ because it is the misfit porosity and, therefore, defined by $\delta\phi_C = \delta\phi - \delta\phi_A - \delta\phi_B$. To see what this quantity must be, consider the general expressions for the change in volume fractions:

$$v_A + \delta v_A = \frac{V_A(1+\nabla\cdot\tilde{\mathbf{u}}_A)}{V_A(1+\nabla\cdot\tilde{\mathbf{u}}_A)+V_B(1+\nabla\cdot\tilde{\mathbf{u}}_B)+V_C+\delta V_C}, \tag{96}$$

$$v_B + \delta v_B = \frac{V_B(1+\nabla\cdot\tilde{\mathbf{u}}_B)}{V_A(1+\nabla\cdot\tilde{\mathbf{u}}_A)+V_B(1+\nabla\cdot\tilde{\mathbf{u}}_B)+V_C+\delta V_C}, \tag{97}$$

and

$$v_C + \delta v_C = \frac{V_C+\delta V_C}{V_A(1+\nabla\cdot\tilde{\mathbf{u}}_A)+V_B(1+\nabla\cdot\tilde{\mathbf{u}}_B)+V_C+\delta V_C}. \tag{98}$$

The resulting formulas for the changes in the porosities (neglecting terms higher than first order are)

$$\delta v_A = v_A[(v_B+v_C)\nabla\cdot\tilde{\mathbf{u}}_A - v_B\nabla\cdot\tilde{\mathbf{u}}_B - \delta V_C/V], \tag{99}$$

$$\delta v_B = v_B[(v_A+v_C)\nabla\cdot\tilde{\mathbf{u}}_B - v_A\nabla\cdot\tilde{\mathbf{u}}_A - \delta V_C/V], \tag{100}$$

and

$$\delta\phi_C = \delta v_C = (1-v_C)\delta V_C/V - v_C(v_A\nabla\cdot\tilde{\mathbf{u}}_A + v_B\nabla\cdot\tilde{\mathbf{u}}_B), \tag{101}$$

satisfying the constraint for volume fractions that $\delta v_A + \delta v_B + \delta v_C = 0$.

Addition of a temperature field does not change the pressure relation (29), but the overall dilatation is now

$$\nabla\cdot\tilde{\mathbf{u}}_s = v_A\nabla\cdot\tilde{\mathbf{u}}_A + v_B\nabla\cdot\tilde{\mathbf{u}}_B + \frac{\delta V_C}{V}, \tag{102}$$

including the added volume introduced by the presence of cracks.

3.2.3. *Definitions of microstructural coefficients*. The micro-pressure laws (30) and (31) must be changed when a temperature field is added because local average pressures are dependent on temperature. Thus, (30) becomes

$$(1-\phi_A)\delta p_A = Q_A(\delta p_c - \gamma_A\delta p_f) + \varepsilon_A\delta T, \tag{103}$$

with a similar expression for the B material. We will see that the Qs and γs have exactly the same algebraic definitions as in the welded case.

In the presence of temperature change, the expressions for response of the dilatations (32)–(33) become

$$-\nabla\cdot\tilde{\mathbf{u}}_A = S_A(\delta p_c - \bar{\alpha}_A\delta p_f) - \bar{\beta}_A\delta T, \tag{104}$$

with a similar relation for material B. We must also add a new relation for the crack dilatation

$$-\frac{\delta V_C}{V} = S_C(\delta p_c - \bar{\alpha}_C \delta p_f) - \bar{\beta}_C \delta T. \tag{105}$$

Cracks can be opened or closed by either pressure or temperature changes, but we assumed that welded contacts prevented this from happening in Section 3.1.

Modified equations for the changes in the porosities are

$$-\delta\phi_A = W_A(\delta p_c - \chi_A \delta p_f) - \kappa_A \delta T, \tag{106}$$

with a similar relation for B. Again, we must add a new law for the cracks

$$-\delta\phi_C = W_C(\delta p_c - \chi_C \delta p_f) - \kappa_C \delta T. \tag{107}$$

This completes the definitions of the microstructural coefficients.

There are 24 coefficients for partially welded or nonwelded contact, compared to 12 for welded contact. Eight of these coefficients are new thermal expansion coefficients, and the other four new ones are related to pressure dependence of the crack phase. The overall strategy used to find these coefficients will be similar to that in the welded case. We will: (1) relate the averaging equations to the temperature-generalized Brown and Korringa laws; (2) introduce the micropressure laws into the A and B Gassmann laws; and (3) use the Berryman and Milton (1992) thought experiment (involving the temperature field this time). It will be possible to express all coefficients exactly in terms of K^*, β^*, β_ϕ and the same known properties of the A and B Gassmann materials used in the welded case.

3.2.4. *General relations for microstructural response.* Now we find from (102) that

$$v_A S_A + v_B S_B + S_C = \frac{1}{K^*}, \tag{108}$$

$$v_A S_A \bar{\alpha}_A + v_B S_B \bar{\alpha}_B + S_C \bar{\alpha}_C = \frac{\alpha}{K^*}, \tag{109}$$

and

$$v_A \bar{\beta}_A + v_B \bar{\beta}_B + \bar{\beta}_C = \beta^*. \tag{110}$$

All three equations follow from the direct generalization of eqn (47). These equations are what allow the crack coefficients S_C, $\bar{\alpha}_C$, and $\bar{\beta}_C$ to be determined.

They also allow us to eliminate the coefficients in (107), by first equating (101)–(107) and then using (108)–(110) to simplify the results to

$$W_C = S_C - \frac{v_C}{K^*}, \tag{111}$$

$$W_C \chi_C = S_C \bar{\alpha}_C - \frac{v_C \alpha}{K^*}, \tag{112}$$

and

$$\kappa_C = \bar{\beta}_C - v_C \beta^*. \tag{113}$$

These three equations are what allow W_C, χ_C and κ_C to be determined.

The addition of the temperature field does not change (29) or (36) and therefore does not change either (37) or (38), with the exception that $v_A + v_B \leqslant 1$ now. Thus, the first two equations for the Qs and γs do not change for nonwelded contact. The temperature dependence on the right-hand side of (29) must average to zero, since the applied pressures are independent variables, so

$$v_A\varepsilon_A+v_B\varepsilon_B=0. \tag{114}$$

Equation (41) changes to

$$-\bar{v}_A\left(\frac{\delta p_A}{K_A}-\beta_A\delta T\right)-\bar{v}_B\left(\frac{\delta p_B}{K_B}-\beta_B\delta T\right)=(1-\phi)\nabla\cdot\bar{\mathbf{u}}_\mathrm{s}-\delta\phi, \tag{115}$$

[some algebra was needed using (99)–(102) to verify that the right-hand side is correct] where now the porosity change is given by

$$-\delta\phi=\frac{\alpha-\phi}{K^*}(\delta p_\mathrm{c}-\delta p_\mathrm{f})+\phi\left(\frac{1}{K_\phi}-\frac{1}{K_\mathrm{s}}\right)\delta p_\mathrm{f}-\phi(\beta_\phi-\beta^*)\delta T. \tag{116}$$

It follows that there is no change in eqns (42) and (43) for the Qs and γs, but that we now have a condition for the micro-pressure temperature coefficients

$$\bar{v}_A\beta_A+\bar{v}_B\beta_B-\frac{v_A\varepsilon_A}{K_A}-\frac{v_B\varepsilon_B}{K_B}=\beta^*-\phi\beta_\phi. \tag{117}$$

The effects of the crack volume (and in fact of all the porosity) have cancelled on the right-hand side of (115), which is why there is no explicit C dependence in either (115) or (117). Solving (114) and (117) for the εs gives

$$v_A\varepsilon_A=\frac{\beta^*-\phi\beta_\phi-\bar{v}_A\Omega_A-\bar{v}_B\beta_B}{1/K_B-1/K_A} \tag{118}$$

$$v_B\varepsilon_B=-v_A\varepsilon_A. \tag{119}$$

Thus, just as in the welded case, we have succeeded in expressing all the micropressure coefficients in terms of the macroscopic moduli and other known properties.

Although not central to our program, we can also make the following observations at this point. Considering the Qs and γs as known, we can return to (93) and (94) to determine the Ws and χs. Substituting (104) and (106) (along with their equivalent B expressions) into (93) and (94) and equating coefficients produces exactly the same eqns (50)–(51) relating the Ws and χs to the Ss and $\bar{\alpha}$s and known quantities. The expressions relating the thermal expansion coefficients turns out to be redundant when compared to others we will obtain next, so we will not show these here. Also, substituting into $\delta\phi=v_A\delta\phi_A+v_B\delta\phi_B+\delta\phi_C+\phi_A\delta v_A+\phi_B\delta v_B$, and equating to (116) just reproduces (108)–(110). All information obtainable from the averaging equations alone has been exhausted.

3.2.5. *Gassmann laws for the* A *and* B *components.* In the presence of an applied temperature change, the micro-confining pressure for Gassmann material A given by $\delta p_{\mathrm{c}A}=(1-\phi_A)\delta p_A+\phi_A\delta p_\mathrm{f}$ is defined

$$\delta p_{\mathrm{c}A}=Q_A\left[\delta p_\mathrm{c}-\frac{(Q_A\gamma_A-\phi_A)}{Q_A}\delta p_\mathrm{f}\right]+\varepsilon_A\delta T \tag{120}$$

with an analagous expression for the B component. The Gassmann laws thus take the form

$$-\nabla\cdot\bar{\mathbf{u}}_A = \frac{Q_A}{K_A^*}\left[\delta p_c - \frac{(\alpha_A-\phi_A+Q_A\gamma_A)}{Q_A}\delta p_f\right] - \left(\beta_A - \frac{\varepsilon_A}{K_A^*}\right)\delta T \tag{121}$$

$$-\delta\phi_A = \frac{(\alpha_A-\phi_A)Q_A}{K_A^*}\left[\delta p_c - \frac{(1-\phi_A+Q_A\gamma_A)}{Q_A}\delta p_f\right] + \frac{(\alpha_A-\phi_A)\varepsilon_A}{K_A^*}\delta T \tag{122}$$

with analagous expressions for the B component.

By comparing to the microstructural laws, we see that because the Qs and γs have the same functional definition as in the welded case, then so do the Ss, Ws, $\bar{\alpha}$s and χs. The temperature coefficients of the microstructural laws are defined by

$$\bar{\beta}_A = \beta_A - \frac{\varepsilon_A}{K_A^*} \tag{123}$$

$$\kappa_A = \frac{(\phi_A-\alpha_A)\varepsilon_A}{K_A^*} \tag{124}$$

with analagous results for the B coefficients.

At this point, just as in the welded case, the only remaining coefficients that must be determined are K_s and K_ϕ (since β_ϕ is assumed known).

3.2.6. *Uniform expansion or contraction for partially welded contact.* In the presence of cracks, it is not possible to guarantee that the confining pressure is uniform throughout the composite porous material (Berryman and Milton, 1992). Therefore, the thought experiment of Berryman and Milton (1991) for welded contact described in the last section is not valid for this case. The key idea behind the success of that approach was that there existed a circumstance in which two independently controlled fields could be used to cause the uniform expansion or contraction of the composite. Having introduced the temperature field (which is assumed here to be uniform throughout the sample given enough time), we can again construct a situation wherein the sample expands or contracts uniformly, but now the two required applied fields are fluid pore pressure and temperature. It is also necessary to set the changes in external confining pressure equal to those of the fluid pressure, so that all differential pressure changes vanish.

Supposing that it is possible to have $\nabla\cdot\bar{\mathbf{u}}_A = \nabla\cdot\bar{\mathbf{u}}_B = \nabla\cdot\bar{\mathbf{u}}_s$, we find the corresponding dilatations are given by

$$\frac{\delta p_f}{K_A} - \beta_A\delta T = \frac{\delta p_f}{K_B} - \beta_B\delta T = \frac{\delta p_f}{K_s} - \beta^*\delta T = -\nabla\cdot\bar{\mathbf{u}}_s. \tag{125}$$

It follows that these equalities are satisfied if the changes of the fields occur in the ratio

$$\frac{\delta T}{\delta p_f} \equiv \tau = \frac{1/K_A - 1/K_B}{\beta_A - \beta_B} \tag{126}$$

$$= \frac{1/K_s - 1/K_A}{\beta^* - \beta_A}\left(= \frac{1/K_s - 1/K_B}{\beta^* - \beta_B}\right), \tag{127}$$

showing that a ratio τ does indeed exist and is expressible in terms of known quantities. Furthermore, we see that K_s has been determined and is related to the known thermal-expansion coefficients. A similar result has been known in the theory of thermoelasticity since the work of Levin (1967) and Cribb (1968), but these authors implicitly assumed welded contact between constituents, and they were not considering porous media.

Changes in fluid pressure and temperature do not induce changes in porosity if the change in differential pressure vanishes while the composite expands or contracts uniformly.

Thus, we obtain another valid relation for the uniform expansion or contraction scenario by setting $\delta\phi = 0$. The result is

$$\frac{1}{K_\phi} = \frac{1}{K_s} + (\beta_\phi - \beta^*)\tau \tag{128}$$

and was obtained earlier by Berryman and Milton (1992). With β_ϕ a known quantity, K_ϕ is determined by this equation. Thus, the entire system of coefficients has been closed at this point and our problem is solved.

One might think that all possible identities have not yet been extracted and that, in particular, β_ϕ might yet be determined (rather than measured). Not only is the macroscopic porosity unchanged when $\delta T/\delta p_f = \tau$, but furthermore all of the component porosity changes vanish ($\delta\phi_A = \delta\phi_B = \delta\phi_C = 0$), and so do the volume fraction changes ($\delta v_A = \delta v_B = \delta v_C = 0$). If we set the left-hand side of (93) equal to (125), we find that

$$-\varepsilon_A\tau = Q_A(1-\gamma_A) - (1-\phi_A), \tag{129}$$

and a similar expression for ε_B holds. The result may be written as

$$v_A\varepsilon_A\tau = \frac{1}{1/K_A - 1/K_B}\left[\frac{\bar{v}_A}{K_A} + \frac{\bar{v}_B}{K_B} - \frac{1}{K_s} + \frac{\phi}{K_\phi}\right]. \tag{130}$$

It is easy to check that these expressions satisfy (114), since changing A to B in (130) only changes the sign of the right hand side. Also, see Appendix B.

Equation (130) shows that (117) becomes

$$\beta^* - \phi\beta_\phi = \bar{v}_A\beta_A + \bar{v}_B\beta_B + \frac{1}{\tau}\left[\frac{1}{K_s} - \frac{\phi}{K_\phi} - \frac{\bar{v}_A}{K_A} - \frac{\bar{v}_B}{K_B}\right], \tag{131}$$

which is a general identity for partially welded, nonwelded, or welded contact, and which may be viewed as another formula relating K_ϕ to β_ϕ. However, (131) is a linear combination of (127) and (128) and thus does not provide a separate equation for β_ϕ. We thus conclude that all possible information has been extracted from this thought experiment and that either β_ϕ or K_ϕ must be measured in order to close the system of coefficients.

3.2.7. *The special case of a rubblized bed.* In special cases, it may be possible to avoid the measurement of β_ϕ. (See also Appendix E.) Consider the extreme case of fully nonwelded contact. By this term, we mean that we have two porous constituents A and B jumbled together so that wherever these two types of materials touch there are no constraints requiring them to remain in contact during a deformation caused by stress or temperature change. This scenario implies that A and B materials form a "rubblized bed" such that even some pieces of A material are not welded to other pieces of A material, and similarly for the B material. This model is special in that results can be obtained for it when only temperature is changed or when only the pore pressure is changed while differential pressure is constant.

Consider the situation when a change in temperature occurs in the fully nonwelded or rubblized material. With no constraints to maintain contact with the surrounding materials, there should be no feedback between temperature changes and average pressure changes in the solid. Thus, we expect to find

$$\varepsilon_A = \varepsilon_B = \kappa_A = \kappa_B = 0. \tag{132}$$

Similarly, the changes in solid volume in this situation are controlled simply by the pure material volume thermal expansion coefficients, so

$$\bar{\beta}_A = \beta_A \quad \text{and} \quad \bar{\beta}_B = \beta_B, \tag{133}$$

which in turn implies that

$$\bar{\beta}_C = \beta^* - v_A\beta_A - v_B\beta_B. \tag{134}$$

And since $\kappa_C = \bar{\beta}_C - v_C\beta^*$ from (113), all the thermal constants are completely determined. In particular, it follows from (117) that

$$\phi\beta_\phi = \beta^* - \bar{v}_A\beta_A - \bar{v}_B\beta_B, \tag{135}$$

showing that the thermal expansion coefficient for the pore space is completely determined from more easily measured quantities in this special case of fully nonwelded contact.

3.2.8. *Summary for partially welded and nonwelded contact.* As was done in the welded case, we now summarize the key results for the case of partially welded and nonwelded contact. The effective stress laws governing the overall response of a multicomponent material in the presence of temperature changes are defined by

$$-\nabla\cdot\bar{\mathbf{u}}_s = \frac{1}{K^*}(\delta p_c - \alpha\delta p_f) - \beta^*\delta T \tag{136}$$

$$-\delta\phi = \frac{\alpha-\phi}{K^*}(\delta p_c - \chi\delta p_f) - \phi(\beta_\phi - \beta^*)\delta T, \tag{137}$$

where the effective-stress coefficients α and χ are algebraically defined as in the welded case. The thermal expansion moduli β^* and β_ϕ are assumed known. For the case of two Gassmann materials in partially welded or nonwelded contact, the moduli K_s and K_ϕ (that are within α and χ) are defined by

$$\frac{1}{K_s} = \frac{1}{K_A} + \tau(\beta^* - \beta_A) \tag{138}$$

$$\frac{1}{K_\phi} = \frac{1}{K_s} + \tau(\beta_\phi - \beta^*). \tag{139}$$

The coefficient

$$\tau = \frac{1/K_A - 1/K_B}{\beta_A - \beta_B} \tag{140}$$

is the special ratio of temperature increment to fluid pressure increment that results in uniform strain.

The microstructural laws are defined by

$$-\nabla\cdot\bar{\mathbf{u}}_A = S_A(\delta p_c - \bar{\alpha}_A\delta p_f) - \bar{\beta}_A\delta T \tag{141}$$

$$-\frac{\delta V_C}{V} = S_C(\delta p_c - \bar{\alpha}_C\delta p_f) - \bar{\beta}_C\delta T \tag{142}$$

$$-\delta\phi_A = W_A(\delta p_c - \chi_A\delta p_f) - \kappa_A\delta T \tag{143}$$

$$-\delta\phi_C = W_C(\delta p_c - \chi_C\delta p_f) - \kappa_C\delta T \tag{144}$$

$$(1-\phi_A)\delta p_A = Q_A(\delta p_c - \gamma_A\delta p_f) + \varepsilon_A\delta T \tag{145}$$

with analagous expressions for the B component. The coefficients S_A, $\bar{\alpha}_A$, W_A, χ_A, Q_A, and γ_A and their corresponding B expressions have the exact algebraic definition given already in the welded case (of course, because K_s and K_ϕ are different from those in the welded case, the numerical values of these coefficients will be different in the two cases). The new coefficients, that are associated with crack openings and the temperature dependence, are defined by

$$v_A\varepsilon_A = \frac{\beta^* - \phi\beta_\phi - \bar{v}_A\beta_A - \bar{v}_B\beta_B}{1/K_B - 1/K_A} \tag{146}$$

$$\bar{\beta}_A = \beta_A - \frac{\varepsilon_A}{K_A^*} \tag{147}$$

$$\bar{\beta}_B = \beta_B - \frac{\varepsilon_B}{K_B^*} \tag{148}$$

$$\bar{\beta}_C = \beta^* - v_A\bar{\beta}_A - v_B\bar{\beta}_B \tag{149}$$

$$S_C = \frac{1}{K^*} - v_AS_A - v_BS_B \tag{150}$$

$$\bar{\alpha}_C = \frac{1}{S_C}\left[\frac{\alpha}{K^*} - v_AS_A\bar{\alpha}_A - v_BS_B\bar{\alpha}_B\right] \tag{151}$$

$$\kappa_C = (1 - v_C)\beta^* - v_A\beta_A - v_B\beta_B \tag{152}$$

$$W_C = \frac{1 - v_C}{K^*} - v_AS_A - v_BS_B \tag{153}$$

$$\chi_C = \frac{1}{W_C}\left[\frac{(1-v)\alpha}{K^*} - v_AS_B\bar{\alpha}_A - v_BS_B\bar{\alpha}_B\right]. \tag{154}$$

Although, in general, β_ϕ is assumed to be a known (measured) constant, for the special case of fully nonwelded contact (a so-called rubblized bed), it takes the special form

$$\phi\beta_\phi = \beta^* - \bar{v}_A\beta_A - \bar{v}_B\beta_B \tag{155}$$

and, thus, need not be measured independently.

4. GENERAL EFFECTIVE STRESS EQUATION FOR MATERIALS WITH WELDED CONTACT

Welded contact may be treated as a special case of partially welded contact. Everything in the analysis goes through, but now $v_C = \delta v_C = \delta\phi_C = 0$, and therefore $S_C = \bar{\alpha}_C = 0$. Both thought experiments considered so far apply simultaneously, so we again have (69) and (70). Thus, we are again in the position of having one eqn (70) in two unknowns (i.e., $\bar{\alpha}_A$ and $\bar{\alpha}_B$).

There is still another legitimate thought experiment for this problem. Since welded contact implies the possibility of constant confining pressure throughout the sample (Berryman and Milton, 1992), we can consider an experiment with $\delta p_f = 0$, and try to find a ratio of δp_c and δT such that $\nabla \cdot \bar{\mathbf{u}}_A = \nabla \cdot \bar{\mathbf{u}}_B = \nabla \cdot \bar{\mathbf{u}}_s$. Then, from (89), we have

$$\frac{\delta p_c}{K_A^*}-\beta_A\delta T=\frac{\delta p_c}{K_B^*}-\beta_B\delta T=\frac{\delta p_c}{K^*}-\beta^*\delta T=\nabla\cdot\bar{\mathbf{u}}_s, \tag{156}$$

showing that

$$\frac{\delta T}{\delta p_c}\equiv\omega=\frac{1/K_A^*-1/K_B^*}{\alpha_A-\beta_B}=\frac{1/K^*-1/K_A^*}{\beta^*-\beta_A}=\frac{1/K^*-1/K_B^*}{\beta^*-\beta_B}. \tag{157}$$

These results are completely equivalent to those of Levin (1967) and Cribb (1968), indicating necessary relations between K^* and β^* for welded contact, but those authors were treating only thermoelastic materials, not thermoporoelastic such as we are treating here.

If we try to carry through the analysis of this thought experiment (even though it is perfectly legitimate), we will learn nothing useful. The reason for this is that the equations resulting from the three thought experiments are not linearly independent. It is easy to see that

$$\frac{\tau}{\omega}=1-\theta. \tag{158}$$

Thus, we cannot obtain any additional constraints on the coefficients this way. Furthermore, we find that the region in which S_A is constant is actually a plane in the three-dimensional space of the field variables and that the functional form of S_A is

$$S_A=S_A[\delta T-\omega(\delta p_c-\theta\delta p_f)], \tag{159}$$

for welded contact. It is easy to check that this is so by writing out the expressions for δv_A and δv_B, and noting that the condition for $\delta v_A=-\delta v_B=0$ is the same as the condition for the argument of (159) to be zero. When the various equalities satisfied by ω, θ, and τ have been accounted for, we find for example that

$$\delta v_A=v_A(\bar{\beta}_A-\beta^*)[\delta T-\omega(\delta p_c-\theta\delta p_f)]. \tag{160}$$

But, no matter how we choose to write this expression, it always has one parameter undetermined, and therefore requires one additional measurement.

It is also important to recognize that we have not exhausted the identities. Considering entropy (which is an extensive quantity), we can obtain another identity in the isothermal thought experiment. The full matrix equation for the thermoporoelastic problem is

$$\begin{pmatrix}\delta e\\-\delta\zeta\\\delta s\end{pmatrix}=\begin{pmatrix}1/K^* & -\alpha/K^* & \beta^*\\-\alpha/K^* & \alpha/K^*+\phi(1/K_f-1/K_\phi) & \phi(\beta_f-\beta_\phi)\\\beta^* & \phi(\beta_f-\beta_\phi) & [\phi c_f+(1-\phi)c_p^*]/T_0\end{pmatrix}\times\begin{pmatrix}-\delta p_c\\-\delta p_f\\\delta T\end{pmatrix}, \tag{161}$$

where the new terms are the change in the entropy δs, and the heat capacities at constant pressure for the fluid c_f and for the composite frame c_p^*. The ambient temperature is T_0. Then, it follows immediately from the isothermal thought experiment with $\delta p_c/\delta p_f=\theta$ that the change in total entropy is the sum of the changes in the two components, so—making use of the bottom row of the matrix—we find

$$\beta^*-\phi\beta_\phi=\bar{v}_A\beta_A+\bar{v}_B\beta_B+(\theta-1)[v_A\beta_A+v_B\beta_B-\beta^*], \tag{162}$$

which is a new identity for β_ϕ that should be compared with the discussion of Appendix B. This formula is true only for materials subject to welded contact between constituents.

5. CONCLUSIONS AND DISCUSSIONS

For the case of an isotropic composite of two Gassmann materials (A and B), we have determined the effective stress laws controlling not only the overall volumetric response but also the response of the individual A and B components. In addition to the basic properties of the individual (isolated) A and B materials, we have expressed the laws using only: (1) the moduli K^* and β^* in the case of fully welded or fully nonwelded contact; and (2) the moduli K^*, β^*, and β_ϕ in the case of partially welded contact. Having such effective stress laws is key to understanding how the physical properties of the earth's crust vary as a function of depth.

The present results are limited by assumptions of isotropy for the constituents, statistical isotropy for the overall poroelastic medium, and by the necessity of assuming the constituents are lumped together as Gassmann materials. Nevertheless, even with these limitations, the subset of real materials covered by this analysis still includes many interesting materials [such as quartz grains embedded in interstitial clay—see Berryman and Milton (1991) for a discussion] that cannot be treated by previous methods. We therefore view the present work as one step toward a more sophisticated analysis that will be required to treat more general porous media.

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APPENDIX A: VOLUME AVERAGING WITH TWO SOLID CONSTITUENTS

When two solid constituents are present, the averaging theorem gives some significantly different results that we will present and discuss here. For simplicity, we will limit the discussion to averages of divergence of displacement and of the displacement itself. We make one assumption implicitly here, that the averaging volume Ω is large enough so that statistical differences between the bulk porosity in the volume and the outcrop of porosity at the surface of the volume are negligible. When this assumption is not made, gradients of porosity appear in the final equations as is shown in Pride and Berryman (1998).

A.1. *Results for all solids*

If there are two solids present as in Fig. 1, for purposes of averaging we may lump them together and act as if they are both "solids", using subscript "s" to refer to the regions occupied by these solids. The averaging theorem (or in this case just the normal divergence theorem) states that

$$\langle \nabla \cdot \mathbf{u}_s \rangle = \frac{1}{V}\int_{\Omega(\mathbf{x})} \nabla \cdot \mathbf{u}_s \, \mathrm{d}^3x = \frac{1}{V}\int_{\partial E} \hat{\mathbf{n}}_s \cdot \mathbf{u}_s \, \mathrm{d}S + \frac{1}{V}\int_{\partial I} \hat{\mathbf{n}}_s \cdot \mathbf{u}_s \, \mathrm{d}S, \tag{163}$$

where Ω is the averaging volume and $V = \int_\Omega \mathrm{d}^3x$, with ∂E being the external boundary and ∂I being the internal or pore boundary. The divergence of the average (found by taking the Frechet derivative with respect to the averaging volume) is

$$\nabla \cdot \langle \mathbf{u}_s \rangle = \frac{1}{V}\int_{\partial E} \hat{\mathbf{n}}_s \cdot \mathbf{u}_s \, \mathrm{d}S. \tag{164}$$

The internal surface integral is easily interpreted as the negative of the change in porosity $\delta\phi$, since the displacement integrated along the bounding surface produces a volume change that is then normalized by the total volume V in the averaging volume $\Omega(\mathbf{x})$, so

$$\frac{1}{V}\int_{\partial I} \hat{\mathbf{n}}_s \cdot \mathbf{u}_s \, \mathrm{d}S = -\delta\phi. \tag{165}$$

The left-hand side of (163) is also easily interpreted as the total change in solid volume $\delta[V(1-\phi)]$ divided by the averaging volume V. Comparing these expressions shows that

$$\nabla \cdot \langle \mathbf{u}_s \rangle = (1-\phi)\frac{\delta V}{V}. \tag{166}$$

Note that there has been no change in the averaging volume Ω, but there has been movement of solid in or out of volume and/or a change in state of compression of the solid. In this regard, our notation is trying to reflect the fact that experimentally we normally start with a fixed volume of material V and then measure changes δV in that volume.

Accounting for the volume occupied by the solid initially, we have $\langle \mathbf{u}_s \rangle \equiv (1-\phi)\bar{\mathbf{u}}_s$, so

$$\nabla \cdot \bar{\mathbf{u}}_s = \frac{\delta V}{V} + \frac{\bar{\mathbf{u}}_s \cdot \nabla\phi}{1-\phi}. \tag{167}$$

We normally neglect the second term on the right-hand side of (167), since we assume that the scales of variation of the displacement field are much smaller than those for the porosity, so that $|\bar{\mathbf{u}}_s \cdot \nabla\phi| \ll |\nabla \cdot \bar{\mathbf{u}}_s|$. However, a fully general analysis must account for the presence of this term [see Pride and Berryman (1998)].

A.2. *Results for constituents* A *and* B *with welded contact*

When we want to distinguish the constituent solids A and B as in Fig. 1, we can break up the averaging volume into two pieces such that

$$\langle \nabla \cdot \mathbf{u}_s \rangle = \langle \nabla \cdot \mathbf{u}_A \rangle + \langle \nabla \cdot \mathbf{u}_B \rangle, \tag{168}$$

which follows immediately from the fact that material A and B occupy disjoint parts of the averaging volume. The averaging theorem for the divergence of $\mathbf{u}_A$ alone then states that

$$\langle \nabla \cdot \mathbf{u}_A \rangle = \nabla \cdot \langle \mathbf{u}_A \rangle + \frac{1}{V}\int_{\partial I_f} \hat{\mathbf{n}}_A \cdot \mathbf{u}_A \, \mathrm{d}S + \frac{1}{V}\int_{\partial I_B} \hat{\mathbf{n}}_A \cdot \mathbf{u}_A \, \mathrm{d}S, \tag{169}$$

where we have explicitly noted that the interior interface has two parts: one boundary ∂I_f with the fluid (or pore space) and one ∂I_B with the other solid (B).

Since

$$\nabla \cdot \langle \mathbf{u}_A \rangle = \frac{1}{V}\int_{\partial E_A} \hat{n}_A \cdot \mathbf{u}_A \, \mathrm{d}S, \tag{170}$$

and a similar expression for $\nabla \cdot \mathbf{u}_B$, we find easily from the identity

$$\frac{1}{V}\int_{\partial E_s} \hat{\mathbf{n}}_s \cdot \mathbf{u}_s \, \mathrm{d}S = \frac{1}{V}\int_{\partial E_A} \hat{\mathbf{n}}_A \cdot \mathbf{u}_A \, \mathrm{d}S + \frac{1}{V}\int_{\partial E_B} \hat{n}_B \cdot \mathbf{u}_B \, \mathrm{d}S \tag{171}$$

that

$$\nabla \cdot \langle \mathbf{u}_s \rangle = \nabla \cdot \langle \mathbf{u}_A \rangle + \nabla \cdot \langle \mathbf{u}_B \rangle. \tag{172}$$

In order to specify the physical significance of $\nabla \cdot \langle \mathbf{u}_A \rangle$, we need to repeat the analysis for all solids, taking into account the fact that when there are two or more solids there must also be additional interior interfaces between these various constituents. Of the four terms in (169), each requires some interpretation. First, the left-hand side has an interpretation similar to that of the left-hand side of (163). Thus, we have the volume average of the dilatation of A material must be

$$\langle \nabla \cdot \mathbf{u}_A \rangle = \frac{\delta[V_A(1-\phi_A)]}{V}, \tag{173}$$

where V_A is the total porous volume of A material and $V_A(1-\phi_A)$ is the total solid volume of A material.

The two integrals on the right-hand side of (169) are more difficult to interpret because they involve the contact region of two Gassmann materials having possibly different porosities. (See Fig. 2.) Statistically the A material should have solid material at this interface occupying the fraction $1-\phi_A$ of the total interface area and A-pore the remaining fraction ϕ_A of the total. The B material has corresponding proportions. Now these continuous surfaces may be statistically correlated or uncorrelated. If they are uncorrelated, we can easily compute the coefficients we will need. But if they are correlated, we introduce new interface constants with the following properties: $\eta_{AB} = \eta_{BA}$ is the fraction of the interface on which solid A touches solid B, η_{Af} is the fraction of the interface on which solid A touches the fluid in B, and similarly η_{Bf} is the fraction on which the solid B touches the fluid in A. With our assumption of statistical homogeneity, these constants should obey the general sum rules $\eta_{AB}+\eta_{Af} = 1-\phi_A$ and $\eta_{BA}+\eta_{Bf} = 1-\phi_B$. One immediate general result is that the difference $\eta_{Af}-\eta_{Bf} = \phi_B-\phi_A$. The solid/solid contact area should be proportional to η_{AB}, which may be very small or it can be as large as the minimum of the two solid fractions $(1-\phi_A)$, $(1-\phi_B)$. For uncorrelated surfaces, we expect $\eta_{AB} = (1-\phi_A)(1-\phi_B)$,

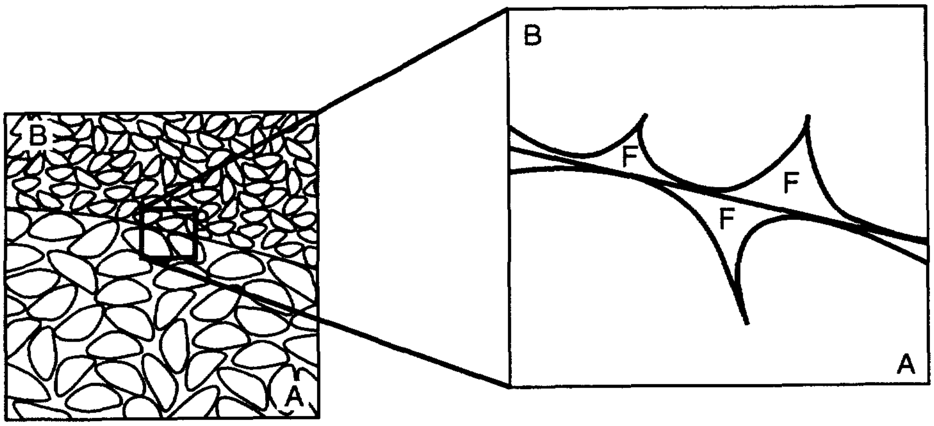

Fig. 2. At an internal interface between Gassmann porous materials A and B, there will be regions where fluid from the A side interfaces with fluid from the B side, where fluid from the A side interfaces with solid from the B side, where solid from the A side interfaces with fluid from the B side, and where solids from both sides touch (not seen in this particular blowup of a two-dimensional cross section of a three-dimensional medium).

$\eta_{Af} = (1-\phi_A)\phi_B$, and $\eta_{Bf} = (1-\phi_B)\phi_A$. These identities are easily shown to satisfy the statistical sum rules for these coefficients. For correlated interfaces, we may view η_{AB} as a new microstructural parameter that characterizes the internal (to the averaging volume) solid/solid interface.

The first integral on the right-hand side of (169) is the surface integral of displacement along the fluid boundary. This term has the same significance as the corresponding one for the whole solid; it is the change in porosity associated with type-A material. The total pore volume associated with A is $V_A\phi_A$, so the change in pore volume must be a change in this quantity. However, the surface integral is strictly over the original boundary of the A material (prior to the displacements $\mathbf{u}_A$), so the correct expression for this change in the absence of other solids is clearly $V_A\delta\phi_A$. But, in the presence of other solids, we must account for the possibility of changes in overall porosity due to changes in volume fraction. Thus, the full contribution of this term is

$$\int_{\partial I_f} \hat{\mathbf{n}}_A \cdot \mathbf{u}_A \, \mathrm{d}S = -V_A\delta\phi_A + V\eta_{Af}\delta v_A, \tag{174}$$

using one of the interface constants introduced in the preceding paragraph. When the volume fraction does not change, as in the case when the averaging volume happens to contain only A material, we see that this expression reduces correctly to (165). When we write the corresponding relation for the B phase and then consider that by definition it must be true that

$$\int_{\partial I_f} \hat{\mathbf{n}}_s \cdot \mathbf{u}_s \, \mathrm{d}S = \int_{\partial I_f} \hat{\mathbf{n}}_A \cdot \mathbf{u}_A \, \mathrm{d}S + \int_{\partial I_f} \hat{\mathbf{n}}_B \cdot \mathbf{u}_B \, \mathrm{d}S, \tag{175}$$

we see that the extra terms proportional to change in volume fraction are exactly what were needed to guarantee that (175) is equivalent to $\delta\phi = v_A\delta\phi_A + v_B\delta\phi_B + (\phi_A - \phi_B)\delta v_A$.

The second integral on the right-hand side of (169) is the surface integral over the AB solid/solid interface. This term also has the important characteristic that it must be exactly the negative of the corresponding term for the B material. So however we interpret it, the expression should be easily identified by the fact that interchanging A and B should change the sign of the term. We make the identification

$$\frac{1}{V}\int_{\partial I_B} \hat{\mathbf{n}}_A \cdot \mathbf{u}_A \, \mathrm{d}S \equiv \eta_{AB}\delta v_A, \tag{176}$$

where δv_A is the change in volume fraction of porous constituent A. This interpretation is reasonable: if a fictitious continuous surface is drawn between the porous constituents and the corresponding surface integral taken, then the result would be exactly δv_A. Since the true AB interface occupies only a fraction η_{AB} of this total interface area, we see that (176) follows.

The remaining term in (169) needing interpretation is proportional to the surface integral of the A component displacement. Combining the previous results, this average must be given by

$$\int_{\partial E_A} \hat{\mathbf{n}}_A \cdot \mathbf{u}_A \mathrm{d}S = (1-\phi_A)[\delta V_A - V\delta v_A] = (1-\phi_A)v_A\delta V, \tag{177}$$

with a matching expression for the B phase. That these two integrals must satisfy the sum rule in (171) together with (166) implies that their sum must be equal to $(1-\phi)\delta V$ which is easily seen to be true (since $v_A + v_B = 1$ and $v_A\phi_A + v_B\phi_B = \phi$) for the case of welded contact.

Comparing all these expressions, we finally determine that

$$\nabla \cdot \langle \mathbf{u}_A \rangle = v(1-\phi_A)\frac{\delta V_A}{V_A} - (1-\phi_A)\delta v_A, \tag{178}$$

which is the desired expression for divergence of the average displacement of A.

Although the divergence of $\langle \mathbf{u}_A \rangle$ is given rigorously by (178), the quantities that actually appear in the quasistatic equations of motion are simply the dilatations of the constituents, so we will define a new quantity $\tilde{\mathbf{u}}_A$ satisfying

$$\nabla \cdot \tilde{\mathbf{u}}_A \equiv \frac{\delta V_A}{V_A}, \tag{179}$$

which is related to $\bar{\mathbf{u}}_A$ by

$$\nabla \cdot \bar{\mathbf{u}}_A = \nabla \cdot \tilde{\mathbf{u}}_A - \frac{\delta v_A}{v_A}, \tag{180}$$

where the change in volume fraction is itself related to $\tilde{\mathbf{u}}_A$ and the corresponding expression of B by $\delta v_A = v_A v_B(\nabla \cdot \tilde{\mathbf{u}}_A - \nabla \cdot \tilde{\mathbf{u}}_B)$. These definitions and interrelations will be important for analysis of wave propagation issues for multicomponent rocks.

APPENDIX B: PRESSURE COEFFICIENT IDENTITY

B.1. *Welded contact*

Using the results from the uniform expansion/contraction thought experiment for confining and fluid pressure changes under isothermal conditions, we can obtain a somewhat simplified expression for the combination of constants [see (44) and (45)]

$$v_A Q_A(1-\gamma_A) = \frac{1}{1/K_B - 1/K_A}\left(\frac{1}{K_B} - \frac{\phi}{K_s} - \frac{1}{K_s} + \frac{\phi}{K_\phi}\right). \tag{181}$$

First recall the identities [see (65) or Berryman and Milton (1991)]

$$\frac{1}{K_s} = \frac{1}{K_A} + (\theta-1)\left(\frac{1}{K_A^*} - \frac{1}{K^*}\right) = \frac{1}{K_B} + (\theta-1)\left(\frac{1}{K_B^*} - \frac{1}{K^*}\right), \tag{182}$$

and

$$\frac{\phi}{K_\phi} - \frac{\phi}{K_s} = (\theta-1)\left[v_A\left(\frac{\alpha_A - \phi_A}{K_A^*}\right) + v_B\left(\frac{\alpha_B - \phi_B}{K_B^*}\right) - \frac{\alpha - \phi}{K^*}\right]. \tag{183}$$

By volume averaging the two expressions in (182), we obtain the intermediate result that

$$\frac{1-\phi}{K_s} = \frac{\bar{v}_A}{K_A} + \frac{\bar{v}_B}{K_B} + (\theta-1)\left(\frac{\bar{v}_A}{K_A^*} + \frac{\bar{v}_B}{K_B^*} - \frac{1-\phi}{K^*}\right). \tag{184}$$

Then, subtracting (183) from (184), we obtain

$$\frac{1}{K_s} - \frac{\phi}{K_\phi} = \frac{\bar{v}_A}{K_A} + \frac{\bar{v}_B}{K_B} + (\theta-1)\left(\frac{v_A}{K_A} + \frac{v_B}{K_B} - \frac{1}{K_s}\right). \tag{185}$$

Substituting (185) into (181) produces the useful identity

$$v_A Q_A(1-\gamma_A) = \bar{v}_A + \frac{\theta-1}{1/K_B - 1/K_A}\left(\frac{1}{K_s} - \frac{v_A}{K_A} - \frac{v_B}{K_B}\right), \tag{186}$$

together with a corresponding result for the *B* component.

B.2. *Partially welded contact*

Using the results from the uniform expansion/contraction thought experiment for temperature and pressure changes at zero differential pressure change, we can also obtain a different simplified expression for the combination of constants in (181). First, recall [see (127) or Berryman and Milton (1992)] that

$$\frac{1}{K_s} = \frac{1}{K_A} + \tau(\beta^* - \beta_A) = \frac{1}{K_B} + \tau(\beta^* - \beta_B), \tag{187}$$

and that

$$\frac{1}{K_\phi} = \frac{1}{K_s} + \tau(\beta_\phi - \beta^*) = \frac{1}{K_A} + \tau(\beta_\phi - \beta_A). \tag{188}$$

The constant τ may be determined by solving for it in the second equation of (187). The second equality in eqn (188) follows by substituting for K_s from (187).

From these equalities, it is straightforward to obtain

$$\frac{1}{K_s} - \frac{\phi}{K_\phi} = \frac{\bar{v}_A}{K_A} + \frac{\bar{v}_B}{K_B} + (\beta^* - \phi\beta_\phi - \bar{v}_A\beta_A - \bar{v}_B\beta_B)\tau. \tag{189}$$

Combining these results with the definition of τ and substituting into (181) produces the final result

$$v_A Q_A(1-\gamma_A) = \bar{v}_A + \frac{1}{\beta_B - \beta_A}[\bar{v}_A\beta_A + \bar{v}_B\beta_B - (\beta^* - \phi\beta_\phi)]. \tag{190}$$

The corresponding result with $A \leftrightarrow B$ also holds. Equation (190) should also be compared to (186).

APPENDIX C: SPECIAL EXAMPLE FOR WELDED CONTACT

In a few cases the algebra for these calculations simplifies. One particularly interesting and easy case assumes that one of the constituents (say *B*) is purely solid, i.e., $\phi_B = 0$ initially and remains so ($\delta\phi_B = 0$). Then, the number of equations that must be solved is dramatically reduced. In particular, we know from the outset that

$K_B^* = K_B$. As long as the other constituent's porosity remains connected, this example is still a valid limit of the equations and has already been considered by Berryman and Milton (1991).

Following the analysis of Section 3.1, it is easily shown that the ratio of confining pressure to pore pressure that gives rise to uniform expansion or contraction is

$$\frac{\delta p_c}{\delta p_f} = \theta = \frac{1/K_A^* - 1/K_A}{1/K_A^* - 1/K_B}. \tag{191}$$

We also find easily that

$$S_B = \frac{Q_B}{K_B} = \frac{K_A}{v_B(K_A - K_B)}\left(\frac{1}{K_s} - \frac{1}{K_A}\right) \tag{192}$$

and that

$$\bar{\alpha}_B = \gamma_B = \phi\left(\frac{1/K_\phi - 1/K_A}{1/K_s - 1/K_A}\right). \tag{193}$$

Substituting these results into (48) and (49), we obtain

$$v_A S_A = \frac{1}{K^*} - \frac{K_A}{K_A - K_B}\left(\frac{1}{K_s} - \frac{1}{K_A}\right), \tag{194}$$

and

$$v_A S_A \bar{\alpha}_A = \frac{\alpha}{K^*} - \frac{\phi K_A}{K_A - K_B}\left(\frac{1}{K_\phi} - \frac{1}{K_A}\right). \tag{195}$$

For porosity variations, $W_B = 0$ and χ_B is arbitrary (and therefore may be taken as zero), while W_A and χ_A are determined by (50) and (51), respectively.

Including the effects of thermal expansion in this model, we fid that the microstructural constants for thermal expansion take the form

$$\bar{\beta}_A = \beta^* + \frac{1}{v_A \tau}\left[\frac{v_B}{K_s} + \frac{K_A}{K_A - K_B}\left(\frac{1-\phi}{K_A} - \frac{1}{K_s} + \frac{\phi}{K_\phi}\right)\right] \tag{196}$$

and

$$\bar{\beta}_B = \beta^* + \frac{1}{v_B \tau}\left[\frac{K_A}{K_A - K_B}\left(\frac{1}{K_s} - \frac{\phi}{K_\phi} - \frac{1-\phi}{K_A}\right) - \frac{v_B}{K_s}\right], \tag{197}$$

which correctly satisfy the constraint $v_A\bar{\beta}_A + v_B\bar{\beta}_B = \beta^*$.

APPENDIX D: EFFECTIVE STRESS RULES FOR MICROSTRUCTURAL PARAMETERS

An implicit assumption in the analysis of welded contact is that during the uniform expansion/contraction experiment the coefficients of the equations for the material dilatations are invariant. So we have

$$-\frac{\delta V_A}{V_A} = S_A(\delta p_c - \bar{\alpha}_A \delta p_f), \tag{198}$$

where S_A and $\bar{\alpha}_A$ are constant along the line $\delta p_c = \theta \delta p_f$, corresponding to uniform expansion or contraction. A similar result holds for the B material. If we want to consider possible deviations from linearity in the overall analysis, we need to maintain the constancy of these coefficients along this line but can suppose that they have δp_c, δp_f dependence of the form

$$S_A = S_A(\delta p_c - \theta \delta p_f) \quad \text{and} \quad \bar{\alpha}_A(\delta p_c - \theta \delta p_f), \tag{199}$$

where θ is a constant. Thus, we are introducing an effective stress rule for these coefficients.

Now it will be useful to consider an integrability condition arising from the fact that

$$S_A \equiv -\left.\frac{\partial \ln V_A}{\partial p_c}\right|_{\delta p_f = 0} \tag{200}$$

and that

$$\bar{\alpha}_A S_A \equiv \left.\frac{\partial \ln V_A}{\partial p_f}\right|_{\delta p_c = 0}. \tag{201}$$

It follows that the cross derivatives must be equal, so

$$\frac{\partial S_A}{\partial p_f} = -\frac{\partial(\bar{\alpha}_A S_A)}{\partial p_c}, \tag{202}$$

this being the standard condition for integrability of such equations.

Using the effective stress rule (199), we find from the chain rule that

$$\frac{\partial S_A}{\partial p_f} = -\theta\frac{\partial S_A}{\partial p_c}. \tag{203}$$

It then follows easily from (202) that

$$(\theta - \bar{\alpha}_A)\frac{\partial S_A}{\partial p_c} = S_A\frac{\partial \bar{\alpha}_A}{\partial p_c}, \tag{204}$$

which may be rewritten as

$$\frac{\partial}{\partial p_c}\ln[(\theta-\bar{\alpha}_A)S_A] = 0. \tag{205}$$

This form of the result shows that

$$S_A = \frac{\text{constant}(p_c)}{\theta - \bar{\alpha}_A}, \tag{206}$$

which should be compared to the welded contact result (69): $S_A = (\theta-\alpha)/K^*(\theta-\bar{\alpha}_A)$. Using effective stress arguments, it is not hard to show that the factor $(\theta-\alpha)/K^*$ is indeed a constant as δp_c varies, as was done previously by Berryman (1992).

APPENDIX E: SPECIAL EXAMPLE FOR PARTIALLY WELDED CONTACT

There exists one example of partially welded contact in which the equations can be completely solved. We consider a granular composite, having two types of solid, nonporous grains labeled A and B. The entire pore phase C consists of the gaps and misfits between and among the grains of both types. The pore phase must be continuous (connected) in order for pore pressure to equilibrate. This model is included in the analysis of the partially welded porous composite by taking the porosities $\phi_A = \phi_B = 0$, and assuming that $\delta\phi_A = \delta\phi_B = 0$ for any state of temperature and stress achieved. The frame moduli satisfy $K_A^* = K_A$ and $K_B^* = K_B$.

For partially welded contact, it is anticipated that the local average pressures experienced by the grains may be influenced by both the confining pressure and the pore pressure. Therefore, we make no assumptions about the values of the $\bar{\alpha}$s and γs.

The values of the coefficients Q and γ are as always given by (44) and (45). Since changes in porosity for constituents A and B cannot occur in this model, $W_A = W_B = 0$, so

$$S_A = \frac{Q_A}{K_A} \tag{207}$$

and

$$\bar{\alpha} = \gamma_A = \phi\frac{1/K_B - 1/K_\phi}{1/K_B - 1/K_s}, \tag{208}$$

and similarly for the B grains. Using (108), we find

$$S_C = \frac{\alpha}{K^*}, \tag{209}$$

while (109) shows that

$$\bar{\alpha}_C = 1 - \frac{\phi K^*}{\alpha K_\phi} \tag{210}$$

follows from (109).

The thermal expansion for the A grains is given by

$$\bar{\beta}_A = \beta^* + \frac{1}{\tau v_A}\left[\frac{K_B}{K_A - K_B}\left(\frac{1-\phi}{K_B} - \frac{1}{K_s} + \frac{\phi}{K_\phi}\right) - \frac{v_A}{K_s}\right], \tag{211}$$

and a similar expression for the B grains [see (197), which happens to be of the same form]. The thermal expansion for the pores then becomes

$$\bar{\beta}_C = v_C\beta^* + \frac{1}{\tau}\left[\frac{\phi}{K_\phi} - \frac{v_C}{K_s}\right]. \tag{212}$$

These formulas also satisfy the constraint $v_A\bar{\beta}_A + v_B\bar{\beta}_B + \bar{\beta}_C = \beta^*$.